MATH 256A: PROBLEM SET #7
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BRIAN OSSERMAN

Exercise 1. (Gluing morphisms II: an open set and a local ring) Let X be a scheme, and $x \in X$ any closed point. Write $U = X \smallsetminus \{x\}$, and $\hat{U} = \operatorname{Spec} \mathcal{O}_{X,x} \smallsetminus \{x\}$ (where here we also write x for the closed point of $\operatorname{Spec} \mathcal{O}_{X,x}$). Then we have a diagram

$$\begin{array}{ccc} \hat{U} & \longrightarrow & \operatorname{Spec} \mathcal{O}_{X,x} \\ \downarrow & & \downarrow \\ U & \longrightarrow & X. \end{array}$$

Let Y be any scheme; then a morphism $f : X \to Y$ yields a pair of morphisms $f_1 : U \to Y$ and $f_2 : \operatorname{Spec} \mathcal{O}_{X,x} \to Y$, which compose to give the same morphism $\hat{U} \to Y$. Show that conversely, given any such f_1, f_2, we have a unique morphism $f : X \to Y$ yielding f_1, f_2.

Exercise 2. (Extending morphisms to proper schemes) Use the valuative criterion of properness to show that if Y is proper over S, and X any scheme over S with a closed point $x \in X$ such that $\operatorname{Spec} \mathcal{O}_{X,x}$ is a valuation ring, then the natural map from morphisms $X \to Y$ over S to morphisms $U \to Y$ over S is a bijection, where $U := X \smallsetminus \{x\}$.

[Note that the condition above that $\mathcal{O}_{X,x}$ be a valuation ring is a strong one. It is satisfied, for instance, if X is a curve over a field and x a smooth point of X, or if $X = \operatorname{Spec} A$ for A a ring of integers.]

Exercise 3. Do Hartshorne, Exercise 5.18 of Chapter II.

Exercise 4. Show that if T is a scheme, $0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0$ a short exact sequence of locally free sheaves on T, and $f : T' \to T$ any morphism, then f^* induces a short exact sequence

$$0 \to f^*\mathcal{F} \to f^*\mathcal{E} \to f^*\mathcal{G} \to 0.$$

Hint: show that the original sequence is locally split.

Recall that we defined the Grassmannian functor as follows: given a scheme S, and a locally free sheaf $\mathcal{E}$ on S of rank d, we define $\underline{G_S(r, \mathcal{E})} : \mathrm{Sch}_S \to \mathrm{Set}$ by

$$\underline{G_S(r, \mathcal{E})}(T) := \{\mathcal{F} \subseteq \mathcal{E}|_T : \mathcal{F} \text{ is locally free of rank } r, \\ \text{and } \mathcal{E}/\mathcal{F} \text{ is locally free of rank } d - r\}.$$

Here $\mathcal{E}|_T$ denotes the pullback of $\mathcal{E}$ from S to T.

Exercise 5. (a) Show that $\underline{G_S(r, \mathcal{E})}$ is a functor, and in fact a Zariski sheaf.

(b) Show that $\underline{G_S(r, \mathcal{E})}$ is representable by a scheme which we denote $G_S(r, \mathcal{E})$, and has a finite open cover by schemes of the form $\mathbb{A}^{r(d-r)}_{S_i}$, where each S_i is an open subscheme of S.

We denote by $G_S(r,d)$ the Grassmannian $G_S(r,\mathcal{O}_S^d)$.

Exercise 6. (a) Show that $G_S(r,d) = G_{\operatorname{Spec}\mathbb{Z}}(r,d) \times_{\operatorname{Spec}\mathbb{Z}} S$.

(b) Use the valuative criterion for properness to show that $G_S(r,d)$ is proper over S.

Exercise 7. Show that $G_S(n,n+1)$ represents the functor asserted in class for $\mathbb{P}^n_S$, sending T to

$$\{(\mathcal{L},(s_0,\dots,s_n)) : \mathcal{L} \text{ is locally free on } T; s_i \in \Gamma(T,\mathcal{L}); \forall t \in T, \exists i \text{ with } s_i \notin \mathfrak{m}_t\mathcal{L}_t\}/\sim .$$

Exercise 8. Let $Z \subseteq X$ be any closed subscheme of a scheme X, and $\mathscr{I}_Z$ the associated ideal sheaf. Then show that on the level of sets, $Z = \{x \in X : (\mathscr{I}_Z)_x \subsetneq \mathcal{O}_{X,x}\}$.

Exercise 9. Let $Y = \operatorname{Spec} k[[t]]$, and let X be the disjoint union over $n \geq 1$ of $\operatorname{Spec} k[[t]]/(t^n)$. Let $\pi : X \to Y$ be the map induced by the quotient maps $k[[t]] \twoheadrightarrow k[[t]]/(t^n)$, and $\pi^\sharp : \mathcal{O}_Y \to \pi_*\mathcal{O}_X$ the map on sheaves. Let $\mathscr{I}$ be the kernel of $\pi^\sharp$.

(a) Show that $\mathscr{I}$ is not quasi-coherent. Conclude that $\pi_*\mathcal{O}_X$ is not quasi-coherent.

(b) What is $Z := \{y \in Y : (\mathscr{I})_y \subsetneq \mathcal{O}_{Y,y}\}$? Show that if $i : Z \to Y$ denotes the inclusion map, and if we set $\mathcal{O}_Z = i^{-1}\mathcal{O}_Y/i^{-1}\mathscr{I}$, then Z has the structure of a locally ringed space, but is not a scheme.