DIMENSION THEORY FOR SCHEMES

BRIAN OSSERMAN

Although dimension theory for arbitrary schemes can be pathological, with appropriate care it is nonetheless possible to make intuitive, dimension-theoretic arguments in situations far more general than the typical case of schemes of finite type over a field. In this note, we briefly survey pathological pitfalls in dimension theory, but mainly focus on positive results, and what hypotheses are necessary in order to insure good behavior. We emphasize the point of view that one should generally work with codimension or local dimension, and that relative dimensions are rather well-behaved for morphisms of finite type.

References are frequently exercises in disguise (or not in disguise, as the case may be). In particular, because the intended applications are geometric, we will state everything in terms of schemes rather than rings. The primary situation which will guide our discussion will be that of a scheme X of finite type over a Noetherian base scheme B, and manipulation of various closed subschemes Z of X. Dimension refers always to Krull dimension, and recall that in this situation, the codimension of Z in X is defined to be the maximal length of a chain of integral closed subschemes containing Z inside X.

1. The Basic Issues

We will work throughout with Noetherian schemes, to maintain some degree of good behavior. If our schemes are of finite type over a field, then dimension is very well-behaved, mainly because it can be understood equivalently in terms of transcendance degree of field extensions. However, we begin by discussing three types of pathologies when one attempts to work more generally: first, that dimension need not be finite; second, that maximal chains of subschemes can have different lengths depending on where they start and finish; and third, that even specifying the ends of such a chain may not determine its length.

An example of Nagata (see Exercise 2 of Problem Set 11) gives a Noetherian (indeed, regular) affine scheme which has infinite dimension, so there is no reasonably general hypothesis which gives finite-dimensionality. Of course, it is true that:

Theorem 1.1. *Noetherian local schemes are finite-dimensional* [1, Cor. 10.7].

and

Theorem 1.2. *Schemes of finite type over a field are finite-dimensional* [1, Easy from Cor. 10.13 a].

Typically, although the Noetherian hypothesis is reasonable for dimension-theory arguments, one does not want to restrict to either local schemes or schemes over a field (let alone those of finite type). One solution would be to add a finite-dimension hypothesis, but as we will see, a cleaner approach is to work with dimensions of local rings, at least on the base B. Another frequently-effective alternative is to substitute codimension for dimension whenever possible; indeed, we will see that these two approaches are often essentially equivalent, although circumstance will often dictate that one is more appropriate than the other.

Slightly more subtle issues of dimension behavior involve how lengths of chain of subschemes behave with certain restrictions. If a scheme has more than one component, they can have unrelated dimensions. Another way of saying this is that in this case, the length of a maximal chain of integral closed subschemes depends on which is the largest subscheme in the chain. This situation is easily avoided, by restricting to irreducible schemes. Luckily, this turns out to be not unreasonably burdensome in practice. However, it turns out that the length of maximal chains can also depend on which closed point is the smallest subscheme of the chain. We can rephrase this by saying that the following is not automatically satisfied (see, e.g. [1, Exer. 13.1]).

$$\dim \mathcal{O}_{X,x} = \dim X \text{ for all } x \in X \text{ closed.} \tag{1.1}$$

It is difficult to impose this condition via extra hypotheses, but working with local rings or codimensions often provides an effective work-around.

More generally, one might very reasonably ask for the following.

$$\operatorname{codim} Z + \dim Z = \dim X. \tag{1.2}$$

Once again, this turns out to be true only very rarely, as the case that Z is a closed point simply recovers (1.1).

However, we observe that if X satisfies the condition that chains between a fixed pair of integral subschemes always have well-behaved lengths, we will have that (1.1) implies (1.2). This condition is not always satisfied, but is a mild hypothesis known as the catenary condition, and is described in the next section.

In effect, we see that dimension is very poorly-behaved globally. On the other hand, as localization won't affect the chains of integral schemes containing a given one, we see:

Proposition 1.3. *Given Z in X an irreducible closed subscheme, and $x \in Z$ any closed point,* $\operatorname{codim} Z$ *is the same in X and in* $\operatorname{Spec} \mathcal{O}_{X,x}$.

As a result, as suggested earlier, one can frequently substitute codimension for dimension in arguments. We will see in concrete situations how this will often suffice in practice, but for the moment we have (with apologies to Körner):

Slogan 1.4. *Dimension is a local property; codimension is a global property.*

2. Catenary Schemes

The technical condition which turns out to be necessary for codimension and dimension to behave manageably is relatively mild, but not automatic.

Definition 2.1. We say that a scheme X is **catenary** if, given any $Z_1 \subseteq Z_2$ integral closed subschemes of X, any two maximal chains of integral closed subschemes of X containing Z_1 and contained in Z_2 have the same length. We say that X is **universally catenary** if X is Noetherian and every scheme Y of finite type over X is catenary.

It is easy to check the following:

Proposition 2.2. *Let X be catenary. Then:*

(i) *if* $Z_1 \subseteq Z_2$ *are irreducible closed subschemes of* X*, then* $\operatorname{codim}_X Z_2 + \operatorname{codim}_{Z_2} Z_1 = \operatorname{codim}_X Z_1$.
(ii) *if* X *is local, all irreducible closed subschemes* Z *of* X *satisfy (1.2);*
(iii) *if* X *satisfies (1.1), then all irreducible closed subschemes* Z *of* X *satisfy (1.2).*

We will see in Theorem 6.1 below that being universally catenary is a rather weak condition, and that in particular any scheme of finite type over a field or any other regular scheme is universally catenary.

3. Equidimensionality

If a scheme X is finite-dimensional, and further (1.2) holds for all closed subschemes of X, we say that X is **equidimensional**. As we have seen, this need not be the case for catenary schemes, nor even for regular, Noetherian schemes over a base field. However, we do have the following positive result:

Proposition 3.1. [5, Prop. 5.2.1] *Let* X *be an irreducible scheme of finite type over a field. Then* X *is equidimensional.*

More generally, as remarked earlier, it is easy to check the following:

Proposition 3.2. *Let* X *be an irreducible catenary scheme satisfying (1.1). Then* X *is equidimensional.*

However, it is difficult to formulate more general conditions under which equidimensionality holds.

Example 3.3. We observe that it is not the case that an irreducible scheme of finite type over a DVR is equidimensional. Indeed, if we consider $X = \operatorname{Spec} R[x]$ for R a DVR with uniformizer t, and take any curve Z in X not having points in the closed fiber, such as $xt = 1$, that Z consists of a single closed point of codimension 1 in X.

We conclude with one case in which we are able to work with schemes which are not irreducible. While irreducible fibers of morphisms of finite type are necessarily equidimensional by the above, the main purpose of the following is to conclude equidimensionality for possibly reducible fibers in open families with irreducible total space. As a bonus, we obtain a generalization of Proposition 3.1.

Theorem 3.4. *Let* X, Y *be irreducible schemes, and* $f : X \to Y$ *an open morphism of finite type. Then for all* $y \in Y$*, the fiber* $f^{-1}y$ *is equidimensional, of dimension equal to* $\dim f^{-1}\eta$*, where* η *is the generic point of* Y*.*

If further f *is closed and* Y *is equidimensional and universally catenary, then* X *is equidimensional.*

Proof. For the first assertion see [6, Cor. 14.2.2]. For the second part, since X will be catenary, we have to show that for any closed point $x \in X$, $\dim \mathcal{O}_{X,x} = \dim X$. First note that because f is closed, any closed point $x \in X$ necessarily lies in a closed fiber. Thus if $y = f(x)$, by equidimensionality of Y we have $\dim \mathcal{O}_{Y,y} = \dim Y$. Further, by our first assertion, and since x is closed, we have $\dim \mathcal{O}_{f^{-1}y,x} = \dim f^{-1}\eta$. Thus by [6, Thm. 14.2.1] or Theorem 5.1 below, $\dim \mathcal{O}_{X,x} = \dim Y + \dim f^{-1}\eta$, and since this is independent of x, we find $\dim \mathcal{O}_{X,x} = \dim X$, as desired. □

4. Local complete intersections

Applying the Krull principal ideal theorem, we saw that if X is an irreducible Noetherian scheme, and Y is defined as the vanishing locus of some n functions f_i on X, then $\operatorname{codim} Y \leq n$. We then defined Y to be a complete intersection if $\operatorname{codim} Y = n$. The basic observation is that this picture works equally well locally:

Proposition 4.1. *Let X be an irreducible Noetherian scheme, and Y a closed subscheme. Suppose that for all $y \in Y$, there exists $f_1, \dots, f_n \in \mathcal{O}_{X,y}$ such that $Y = V(f_1, \dots, f_n)$ locally at y (i.e., $\mathcal{O}_{Y,y} \cong \mathcal{O}_{X,y}/(f_1, \dots, f_n)$). They every component of Y has codimension at most n in X.*

This leads us to define:

Definition 4.2. Let X be an irreducible Noetherian scheme, and Y a closed subscheme. Suppose that for some n, every component Z of Y has codimension n in X, and that for all $y \in Y$, there exists $f_1, \dots, f_n$ in $\mathcal{O}_{X,y}$ such that $\mathcal{O}_{Y,y} \cong \mathcal{O}_{X,y}/(f_1, \dots, f_n)$. Then we say that Y is **local complete intersection** inside X.

Our immediate application of local complete intersections involves the following basic proposition:

Proposition 4.3. *Let X be a regular scheme, and Y a regular closed subscheme of X. Then Y is a local complete intersection inside X.*

Proof. See Exercise 3 of Problem Set 11. □

More generally, the situation that X is regular is important. It turns out that in this case, Y is rather well-behaved, and that moreover, whether Y is a local complete intersection in X is independent of X. In this case, we say that Y is a **local complete intersection** scheme. See [3, Thm. 21.2].

5. Relative dimensions

Finally, we review certain positive results which allow one to compute (co)dimensions in practice.

Theorem 5.1. *Let $f : X \to Y$ be a morphism and $x \in X$ any point. Then*

$$\dim \mathcal{O}_{X,x} \leq \dim \mathcal{O}_{Y,f(x)} + \dim \mathcal{O}_{f^{-1}(f(x)),x},$$

with equality holding if f is open.

Proof. See [1, Thm. 10.10] for the inequality, and equality in the case that f is flat; the more general statement is [6, Thm. 14.2.1]. □

Theorem 5.2. *Let $f : X \to Y$ be a morphism of irreducible schemes, with Y regular, and Z an irreducible closed subscheme of Y. Then any irreducible component Z' of $f^{-1}(Z)$ has codimension in X at most equal to the codimension of Z in Y.*

Proof. In fact, Hochster shows [2, Thm. 7.1] that this result follows from the case that $X \to Y$ is a closed immersion, which is the following (deep) theorem of Serre. □

Theorem 5.3. *Let Z_1, Z_2 be irreducible subschemes of an irreducible regular scheme X, and Z_3 any irreducible component of $Z_1 \cap Z_2$. Then $\operatorname{codim} Z_3 \leq \operatorname{codim} Z_1 + \operatorname{codim} Z_2$.*

Proof. See [4, Thm. V.3] for the statement in terms of local rings. □

Remark 5.4. There is a special case in which the above results are much easier to prove, but which requires the notion of smoothness. See Proposition 6.2 below.

Example 5.5. Note that the regularity is a vital hypothesis for the above to be true: if X is a cone over a quadric surface, and Z_1, Z_2 are cones corresponding to two distinct lines in one ruling of the surface, then each has codimension 1, but their intersection is only at the cone point, which has codimension 3.

Theorem 5.6. *Let $f : X \to Y$ be a morphism locally of finite type. For any $n \geq 0$, the locus $F_n(X) := \{x \in X : \dim_x(f^{-1}(f(x))) \geq n\}$ is closed in X, where $\dim_x$ denotes the dimension of any components meeting x.*

In particular, if also f is closed, the locus $F_n(Y) := \{y \in Y : \dim f^{-1}(y) \geq n\}$ is closed.

Proof. See [7, Thm. 13.1.3], [7, Cor. 13.1.5]. □

6. Additional results

We include here some results and arguments which require some yet-to-be-defined concepts, namely smoothness and Cohen-Macaulayness.

We first have a very strong criterion for being universally catenary.

Theorem 6.1. [3, Thm. 17.9] *Let X be of finite type over a locally Cohen-Macaulay scheme. Then X is universally catenary.*

Next, we can give a simple argument for Theorem 5.2 in the case that Y is smooth over a field k:

Proposition 6.2. *Let $f : X \to Y$ be a morphism of geometrically irreducible schemes of finite type over a field k, with Y smooth over k, and Z a geometrically irreducible closed subscheme of Y. Then any irreducible component Z' of $f^{-1}(Z)$ has codimension in X at most equal to the codimension of Z in Y.*

Proof. The basic observation is that because Y is smooth over k, we have that Y and $Y \times_k Y$ are regular, so that the diagonal $\Delta_Y \subseteq Y \times_k Y$ is a local complete intersection, of codimension $\dim Y$. Then if $i : Z \hookrightarrow Y$ is the inclusion, we can write $f^{-1}(Z)$ as the fiber product of $f \times i : X \times_k Z \to Y \times_k Y$ with the diagonal $\Delta_Y : Y \to Y \times_k Y$. We therefore have that $f^{-1}(Z)$ is locally cut out by $\dim Y$ equations inside $X \times_k Z$, which is irreducible of dimension $\dim X + \dim Z$. Hence, every component of $f^{-1}(Z)$ has dimension at least $\dim X + \dim Z - \dim Y$, and hence codimension in X at most $\dim Y - \dim Z$, as desired. □

References

1. David Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Springer-Verlag, 1995.
2. Melvin Hochster, *Big cohen-macaulay modules and algebras and embeddability in rings of witt vectors*, Queen's Papers on Pure and Applied Math **42** (1975), 106–195.
3. Hideyuki Matsumura, *Commutative ring theory*, Cambridge University Press, 1986.
4. J. P. Serre, *Algèbre locale. multiplicités*, Lecture Notes in Mathematics, no. 11, Springer-Verlag, 1965.
5. Alexander Grothendieck with Jean Dieudonné, *Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas, seconde partie*, vol. 24, Publications mathématiques de l'I.H.É.S., no. 2, Institut des Hautes Études Scientifiques, 1965.

6. ______, *Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas, troisième partie*, vol. 28, Publications mathématiques de l'I.H.É.S., no. 2, Institut des Hautes Études Scientifiques, 1966.
7. ______, *Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas, quatriéme partie*, vol. 32, Publications mathématiques de l'I.H.É.S., no. 2, Institut des Hautes Études Scientifiques, 1967.