MATH H113 PROBLEM SET 6
DUE 3/17

For this problem set, all rings R are assumed to be integral domains.

(1) R is said to be a **Euclidean domain** if there exists a function $N : R \to \mathbb{Z}_{\geqslant 0}$ such that:

(i) $N(r) = 0$ if and only if $r = 0$;

(ii) For any $a, b \in R$ with $b \neq 0$, there exist $q, r \in R$ such that $a = bq + r$ and $N(r) < N(b)$.

Prove that if R is a Euclidean domain, then R is a principal ideal domain.

(2) We say that an element $p \in R$ is **prime** if if pR is a non-zero prime ideal (i.e., it is a prime ideal which is not just $\{0\}$). We say p is **irreducible** if for all $x, y \in R$ such that $p = xy$, exactly one of x and y is in $R^\times$.

(a) Show that if p is prime, then p is irreducible.

(b) Show that if R is a PID, and p is irreducible, then p is prime.

Remark: the condition that every irreducible element is prime is very closely related to the condition that in R, every element can be factored uniquely as a product of irreducible (prime) elements. This property is arguably the most important consequence of being a PID.

(3) Show that $\mathbb{Z}[i]$ is a Euclidean domain, with $N(a + bi)$ defined to be $a^2 + b^2$.

(4) Show that $k[x]$ is a Euclidean domain where k is any field, with $N(f(x))$ defined to be $\deg f + 1$ for non-zero polynomials, and $N(0) = 0$.

(5) Show that $\mathbb{Z}[\sqrt{-3}] := \{a + b\sqrt{-3} : a, b \in \mathbb{Z}\}$ is not a Euclidean domain by producing an element which is irreducible but not prime.