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# Correlation Functions of Random Matrix Ensembles Related to Classical Orthogonal Polynomials

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In the theory of level statistics, the statistical properties of energy levels are obtained from the correlation functions of random matrix ensembles. A class of matrix ensembles, which are related to classical orthogonal polynomials, has extensively been investigated in the case of complex hermitian random matrices. We systematically evaluate the correlation functions of the random matrix ensembles in all the three cases of complex hermitian, real symmetric and self-dual quaternion random matrices.

## §1. Introduction

The statistical theory of complex spectra has been developed in order to account for properties of highly excited states of nuclear systems.[1-4] Recently it has drawn a renewed interest because it is expected to characterize quantum mechanics of not only complex but also simple systems when the underlying classical systems are nonintegrable.[5,6] The statistical properties of energy levels can be obtained from the $k$-level correlation functions defined as

$$I_{N\beta}^{(k)}(x_1, \cdots, x_k)=\frac{1}{(N-k)!}\int\cdots\int P_{N\beta}(x_1, \cdots, x_N)\,\mathrm{d}x_{k+1}\cdots \mathrm{d}x_N, \tag{1.1}$$

where $P_{N\beta}(x_1,\cdots,x_N)$ is the joint probability distribution function of $N$ eigenvalues $x_1, \cdots, x_N$ of a random matrix chosen from an ensemble of $N\times N$ Hermitian matrices.

When we study a random matrix ensemble, it is customary to evaluate the level density and the level spacing distribution and to compare them with experimental data. The level density is just the limit of the 1-level correlation function $I_{N\beta}^{(1)}$ and is one of the global $k$-level correlation functions which are defined as

$$G_\beta^{(k)}(x_1, \cdots, x_k)\equiv\lim_{N\to\infty}\,[I_{N\beta}^{(k)}(x_1, \cdots, x_k)/I_{N\beta}^{(0)}]. \tag{1.2}$$

On the other hand, the level spacing distribution must be considered as a local statistical behavior. That is, the spacings are measured in a region $\delta x$ which is narrow enough to have an almost constant level density. Then the spacings are expressed in units of the mean spacing for this narrow region. The local statistical behaviors at $x=w$ are obtalned from the local $k$-level correlation functions which are defined as

$$L_\beta^{(k)}(w;\xi_1, \cdots, \xi_k)\equiv\lim_{N\to\infty}\left(\frac{1}{G_\beta^{(1)}(w)}\right)^k[I_{N\beta}^{(k)}(x_1, \cdots, x_k)/I_{N\beta}^{(0)}], \tag{1.3}$$

where

$$x_i=w+\frac{1}{G_\beta^{(1)}(w)}\xi_i.$$

Various random matrix ensembles are proposed and three of them have extensively been studied: Gaussian ensembles, circular ensembles and Legendre ensembles. The arguments of Gaussian ensembles are based upon the following two physical postulates.

(1) The ensemble is statistically invariant under the change of basis.

(2) The matrix elements are statistically independent.

If we admit these two assumptions, the Hamiltonian matrix ensemble is reduced to Gaussian ensembles of the eigenvalues. Since any particular representation should not play a particular role in the Hamiltonian statistics, the first requirement is quite natural. However the second assumption seems to be an over-restriction which is artificially introduced to simplify the calculations.

Circular ensembles were introduced by Dyson[7] and can be considered as ensembles of classical one-component Coulomb gas on a circle. They are mathematically simpler than Gaussian ensembles. The local statistical properties of them are identical to those of Gaussian ensembles.

Legendre ensembles have been studied by Vo-Dai and Derome[8] and by Mehta.[9] Their local statistical behaviors have been found identical to those of Gaussian and circular ensembles.

When we compare the consequences from a random matrix model with experiments, we must note that the underlying space and time symmetries possessed by the system impose restrictions on admissible matrix ensembles. If a system is invariant under time reversal and rotations, the Hamiltonian matrix is real symmetric. A system with odd spin, invariant under time reversal but having no rotational symmetry, corresponds to a self-dual quaternion matrix. A system without time-reversal invariance has a Hamiltonian that may be an arbitrary complex hermitian matrix not restricted to be real or self-dual. Systems of the first type are observed in nuclear physics, and therefore real symmetric Hamiltonians are physically most important. Recently, systems of the third type have been studied by numerical experiments on simple model Hamiltonians.[10,11]

As for global distributions, Gaussian, circular and Legendre ensembles show different behaviors neither of which is consistent with the properties of real nuclei or model Hamiltonians. However, their local statistical properties agree with experiments. This implies that there exist certain universal aspects among random matrix models which well describe the fundamental properties of nonintegrable quantum mechanical systems.

Being free from severe physical requirements, we are able to introduce infinite number of random matrix models. Gaussian ensembles and Legendre ensembles are related to the Hermite polynomials and the Legendre polynomials respectively and other random matrix models which are related to other orthogonal polynomials can be also formulated. Since we wish to understand the universal behaviors, it is desirable to investigate the models which are related to general orthogonal polynomials.

In this paper we deal with ensembles related to what is called classical orthogonal polynomials. It seems preferable to restrict our attention to them because at present the theory of classical orthogonal polynomials provides us the most powerful method. These ensembles have been studied by several authors in the case of complex hermitian random matrices.[12-14] Therefore we like to focus on the other two cases.

In the cases of real symmetric and self-dual quaternion random matrices, we must work with skew orthogonal polynomials which are not so familiar as ordinary orthogonal polynomials. We explicitly present skew orthogonal polynomials and their normalization constants which are related to classical orthogonal polynomials. Further, we illustrate how the correlation functions can be derived from them.

The outline of this paper is as follows. In §2, we present various ensembles related to classical orthogonal polynomials. In §3, we summarize Dyson's quaternion matrix method for the calculation of correlation functions. In §4, we illustrate the calculation of the global and the local correlation functions in the case of complex hermitian random matrices. In §5, we evaluate the correlation functions in the case of real symmetric random matrices. In §6, the case of self-dual quaternion matrices are treated. The last section is devoted to discussions on general case.

## §2. Random Matrix Ensembles Related to Classical Orthogonal Polynomials

It is known that the $N$ eigenvalue distributions $\{x_i\}$ of random matrices are reduced to the thermodynamic distributions of logarithmically interacting $N$ particles on a straight line.[4] The Hamiltonian $H$ is

$$H=\sum_{i=1}^{N} V(x_i)-\sum_{i<j}^{N}\log|x_i-x_j|,$$
$$-\infty<x_1,\cdots,x_N<\infty. \tag{2.1}$$

The first and the second terms denote constraining one-body potential and logarithmic repulsive two-body potential, respectively. Thus the joint probability distribution function of $N$ eigenvalues $x_1,\cdots,x_N$ is given by

$$P_{N\beta}(x_1,\cdots,x_N)=\mathrm{e}^{-\beta H}. \tag{2.2}$$

The cases $\beta=1$, 2 and 4 correspond to real symmetric, complex hermitian and self-dual quaternion random matrices, respectively.

In this paper we are concerned with ensembles with the following special one-body potentials:

(1) Jacobi ensembles

$$\begin{aligned} V(x_i)&=-a\log(1-x_i)-b\log(1+x_i), && -1<x_i<1,\\ &=\infty, && \text{otherwise.}\end{aligned} \tag{2.3}$$

(2) Legendre ensembles

$$\begin{aligned} V(x_i)&=0, && -1<x_i<1,\\ &=\infty, && \text{otherwise.}\end{aligned} \tag{2.4}$$

(3) Gaussian ensembles

$$V(x_i)=\frac{x_i^2}{2}. \quad -\infty<x_i<\infty. \tag{2.5}$$

(4) Laguerre ensembles

$$\begin{aligned} V(x_i)&=-a\log x_i+x_i, && 0<x_i<\infty,\\ &=\infty, && \text{otherwise.}\end{aligned} \tag{2.6}$$

Here $a$, $b\geq 0$. The Boltzmann weights for these one-body potentials take the forms of the orthogonality weight functions for classical orthogonal polynomials. The definitions and the orthogonality relations of classical orthogonal polynomials are summarized as follows.[15]

(1) Jacobi polynomials

$$P_n^{(\lambda,\mu)}(x)\equiv\frac{1}{(1-x)^{\lambda}(1+x)^{\mu}}\frac{(-1)^n}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\{(1-x)^{n+\lambda}(1+x)^{n+\mu}\}, \tag{2.7}$$

$$\int_{-1}^{1}P_m^{(\lambda,\mu)}(x)P_n^{(\lambda,\mu)}(x)(1-x)^{\lambda}(1+x)^{\mu}\,\mathrm{d}x=2^{\lambda+\mu+1}\frac{\Gamma(\lambda+1)\Gamma(\mu+1)}{\Gamma(\lambda+\mu+2)}\delta_{mn}. \tag{2.8}$$

(2) Legendre polynomials are special cases of Jacobi polynomials, $P_n^{(0,0)}(x)$.

(3) Hermite polynomials

$$H_n(x)\equiv(-1)^n\mathrm{e}^{x^2}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(\mathrm{e}^{-x^2}), \tag{2.9}$$

$$\int_{-\infty}^{\infty}H_m(x)H_n(x)\mathrm{e}^{-x^2}\mathrm{d}x=\sqrt{\pi}\,2^n n!\,\delta_{mn}. \tag{2.10}$$

(4) Laguerre pokynomials

$$L_n^{(\lambda)}(x)\equiv\frac{x^{-\lambda}\mathrm{e}^{x}}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(\mathrm{e}^{-x}x^{n+\lambda}), \tag{2.11}$$

$$\int_0^\infty L_m^{(\lambda)}(x)L_n^{(\lambda)}(x)x^\lambda e^{-x}dx=\frac{\Gamma(n+\lambda+1)}{\Gamma(n+1)}\delta_{mn}. \tag{2.12}$$

Here $\Gamma(z)$ is the gamma function.

The partition functions $I_{N\beta}^{(0)}$ for these ensembles have been derived[4,16] from Selberg's integral formula[17]:

(1) Jacobi ensembles

$$I_{N\beta}^{(0)}=\frac{1}{N!}\int_{-1}^{1}\cdots\int_{-1}^{1}\prod_{i=1}^{N}(1-x_i)^{a\beta}(1+x_i)^{b\beta}\prod_{i<j}^{N}|x_i-x_j|^\beta dx_1\cdots dx_N$$
$$=\frac{2^{\frac{\beta N(N-1)}{2}+N(a\beta+b\beta+1)}}{N!}\prod_{j=0}^{N-1}\frac{\Gamma\left(1+\frac{\beta}{2}+\frac{j\beta}{2}\right)\Gamma\left(a\beta+\frac{j\beta}{2}+1\right)\Gamma\left(b\beta+\frac{j\beta}{2}+1\right)}{\Gamma\left(1+\frac{\beta}{2}\right)\Gamma\left(a\beta+b\beta+2+\frac{\beta(N+j-1)}{2}\right)}. \tag{2.13}$$

(2) Legendre ensembles

$$I_{N\beta}^{(0)}=\frac{1}{N!}\int_{-1}^{1}\cdots\int_{-1}^{1}\prod_{i<j}^{N}|x_i-x_j|^\beta dx_1\cdots dx_N$$
$$=\frac{2^{\beta N(N-1)/2+N}}{N!}\prod_{j=0}^{N-1}\frac{\Gamma(1+\beta/2+j\beta/2)[\Gamma(j\beta/2+1)]^2}{\Gamma(1+\beta/2)\Gamma(2+\beta(N+j-1)/2)}. \tag{2.14}$$

(3) Gaussian ensembles

$$I_{N\beta}^{(0)}=\frac{1}{N!}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{i=1}^{N}e^{-\frac{\beta}{2}x_i^2}\prod_{i<j}^{N}|x_i-x_j|^\beta dx_1\cdots dx_N$$
$$=\frac{(2\pi)^{N/2}\beta^{-N(\beta(N-1)/2+1)/2}}{N!}\prod_{j=1}^{N}\frac{\Gamma(1+j\beta/2)}{\Gamma(1+\beta/2)}. \tag{2.15}$$

(4) Laguerre ensembles

$$I_{N\beta}^{(0)}=\frac{1}{N!}\int_0^\infty\cdots\int_0^\infty\prod_{i=1}^{N}x_i^{a\beta}e^{-\beta x_i}\prod_{i<j}^{N}|x_i-x_j|^\beta dx_1\cdots dx_N$$
$$=\frac{\beta^{-\beta N(N-1)/2-a\beta N-N}}{N!}\prod_{j=0}^{N-1}\frac{\Gamma(1+\beta/2+j\beta/2)\Gamma(a\beta+j\beta/2+1)}{\Gamma(1+\beta/2)}. \tag{2.16}$$

In addition, the free energies have explicitly been evaluated in the thermodynamic limit.[16]

## §3. Quaternion Matrix Method

In this section, we shall illustrate Dyson's quaternion matrix method to evaluate the $k$-level correlation functions.[18-20] A quaternion is a linear combination of four basic units $\{1, e_1, e_2, e_3\}$:

$$q=q_0+\boldsymbol{q}\cdot\boldsymbol{e}=q_0+q_1e_1+q_2e_2+q_3e_3, \tag{3.1}$$

where $q_0$, $q_1$, $q_2$ and $q_3$ are real or complex numbers. In (3.1), $q_0$ is called the scalar part of $q$. The four basic units satisfy the multiplication laws

$$1\cdot1=1,\quad 1\cdot e_j=e_j\cdot1=e_j,\quad j=1,2,3, \tag{3.2}$$

$$e_1^2=e_2^2=e_3^2=e_1e_2e_3=-1, \tag{3.3}$$

and multiplication is associative. Any quaternion has the dual

$$\bar{q}=q_0-\boldsymbol{q}\cdot\boldsymbol{e}. \tag{3.4}$$

A matrix $Q$ with quaternion elements $q_{ij}$ has a dual matrix $\bar{Q}=[\bar{q}_{ji}]$. The quaternion units can be considered as $2\times2$ matrices with complex elements

$$1\to\begin{bmatrix}1&0\\0&1\end{bmatrix},\quad e_1\to\begin{bmatrix}0&-1\\1&0\end{bmatrix},$$
$$e_2\to\begin{bmatrix}0&-i\\-i&0\end{bmatrix},\quad e_3\to\begin{bmatrix}i&0\\0&-i\end{bmatrix}. \tag{3.5}$$

We define a determinant Tdet of a self-dual $Q$(i.e., $Q=\bar{Q}$) as[21,22)]

$$\mathrm{T}\det Q=\sum_P (-1)^{N-l}\prod_1^l (q_{ab}q_{bc}\cdots q_{da})_0, \tag{3.6}$$

where $P$ is any permutation of the indices $(1, 2,\cdots, N)$ consisting of $l$ exclusive cycles of the form $(a\to b\to c\to\cdots\to d\to a)$ and $(-1)^{N-l}$ is the parity of $P$. The subscript 0 means that we take the scalar part of the product over each cycle. We have two theorems which will be useful in the subsequent sections.

*Theorem 1*: Replacing the quaternion elements of a self-dual matrix $Q$ by their $2\times 2$ representatives, we get a matrix $C(Q)$ of twice the size with complex elements. Then

$$\det C(Q)=\det C(\bar{Q})=(\mathrm{T}\det Q)^2. \tag{3.7}$$

*Theorem 2*: Let the quaternion elements $q_{ij}$ of a self-dual $N\times N$ matrix $Q_N$ depend on $N$ real or complex variables $x_1, x_2, \cdots, x_N$, as

$$q_{ij}=f(x_i, x_j). \tag{3.8}$$

We assume that $f(x, y)$ satisfies the following conditions.

$$\int f(x, x)\,\mathrm{d}\mu(x)=c, \tag{3.9}$$

$$\int f(x, y)f(y, z)\,\mathrm{d}\mu(y)=f(x, z)+g(x, z), \tag{3.10}$$

$$g(x, y)=\lambda f(x, y)-f(x, y)\lambda, \tag{3.11}$$

where $\mathrm{d}\mu(x)$ is a suitable measure, $c$ a constant scalar, and $\lambda$ a constant quaternion. Then

$$\int \mathrm{T}\det Q_N\,\mathrm{d}\mu(x)=(c-N+1)\,\mathrm{T}\det Q_{N-1}, \tag{3.12}$$

where $Q_{N-1}$ is the $(N-1)\times(N-1)$ matrix obtained by removing the row and the column containing $x_N$.

## §4. Complex Hermitian Random Matrices

In this section, we shall illustrate the derivation of the global and local correlation functions for the case $\beta=2$.

Let us introduce monic orthogonal polynomials

$$C_n(x)=x^n+\cdots, \tag{4.1}$$

which satisfy

$$\int \mathrm{e}^{-2V(x)}C_m(x)C_n(x)\,\mathrm{d}x=h_m\delta_{mn}. \tag{4.2}$$

We also introduce $N\times N$ quaternion matrices $M$ and $[f_2(x_i, x_j)]_N$ whose elements have only scalar parts

$$M_{ij}=\mathrm{e}^{-V(x_i)}h_{j-1}^{-1/2}C_{j-1}(x_i), \tag{4.3}$$

$$f_2(x_i, x_j)=\mathrm{e}^{-[V(x_i)+V(x_j)]}\sum_{m=0}^{N-1}h_m^{-1}C_m(x_i)C_m(x_j). \tag{4.4}$$

Then we can show the following identity.

$$\prod_{i=1}^N \mathrm{e}^{-2V(x_i)}\prod_{i<j}^N |x_i-x_j|^2=\left(\prod_{m=0}^{N-1}h_m\right)\mathrm{T}\det(M\bar{M})=\left(\prod_{m=0}^{N-1}h_m\right)\mathrm{T}\det[f_2(x_i, x_j)]_N. \tag{4.5}$$

It is easy to see that $f_2(x, y)$ satisfies

$$\bar{f}_2(y, x)=f_2(x, y), \tag{4.6}$$

$$\int f_2(x, x)\,\mathrm{d}x=N, \tag{4.7}$$

and

$$\int f_2(x, y)f_2(y, z)\,\mathrm{d}y=f_2(x, z). \tag{4.8}$$

Therefore we can apply *Theorem 2*. The $k$-level correlation functions (1.1) and the partition function, respectively, are expressed by

$$I_{N2}^{(k)}(x_1, x_2, \cdots, x_k)=\frac{1}{(N-k)!}\int \prod_{i=1}^{N} e^{-2V(x_i)} \prod_{i<j}^{N} |x_i-x_j|^2 \, dx_{k+1} \cdots dx_N$$
$$=\frac{1}{(N-k)!}\left(\prod_{m=0}^{N-1} h_m\right)\int \mathrm{T}\det[f_2(x_i, x_j)]_N \, dx_{k+1} \cdots dx_N$$
$$=\left(\prod_{m=0}^{N-1} h_m\right) \mathrm{T}\det[f_2(x_i, x_j)]_k, \tag{4.9}$$

$$I_{N2}^{(0)}=\prod_{m=0}^{N-1} h_m. \tag{4.10}$$

Thus if $h_m$'s and $f_2(x, y)$ are given, all the thermodynamic quantities are completely determined. The problem is reduced to the asymptotic evaluations in the limit $N\to\infty$ of the Christoffel-Darboux formula[15)]

$$f_2(x, y)=e^{-[V(x)+V(y)]}\sum_{m=0}^{N-1}\frac{1}{h_m} C_m(x) C_m(y)$$
$$=e^{-[V(x)+V(y)]}\frac{1}{h_{N-1}}\frac{C_N(x) C_{N-1}(y)-C_{N-1}(x) C_N(y)}{x-y}, \tag{4.11}$$

and a confluent form of the Christoffel-Darboux formula

$$f_2(x, x)=e^{-2V(x)}\sum_{m=0}^{N-1}\frac{1}{h_m}\{C_m(x)\}^2=e^{-2V(x)}\frac{1}{h_{N-1}}\{C_N'(x) C_{N-1}(x)-C_{N-1}'(x) C_N(x)\}. \tag{4.12}$$

(1) The Jacobi ensemble

From eqs. (2.7) and (2.8), the monic Jacobi polynomials are given by

$$C_n(x)=2^n n!\frac{\Gamma(2a+2b+n+1)}{\Gamma(2a+2b+2n+1)} P_n^{(2a,2b)}(x), \tag{4.13}$$

and satisfy the orthogonality relation

$$\int e^{-2V(x)} C_m(x) C_n(x)\, dx=\int_{-1}^{1}(1-x)^{2a}(1+x)^{2b} C_m(x) C_n(x)\, dx$$
$$=2^{2n+2a+2b+1} n!\frac{\Gamma(2a+n+1)\Gamma(2b+n+1)\Gamma(2a+2b+n+1)}{\Gamma(2a+2b+2n+1)\Gamma(2a+2b+2n+2)}\delta_{mn}$$
$$=h_n\delta_{mn}. \tag{4.14}$$

The asymptotic forms of the monic Jacobi polynomials are known to be [15)]

$$C_n(\cos\theta)=2^n n!\frac{\Gamma(2a+2b+n+1)}{\Gamma(2a+2b+2n+1)}$$
$$\times\left[\frac{\cos\left\{\left(n+\dfrac{2a+2b+1}{2}\right)\theta-\left(2a+\dfrac{1}{2}\right)\dfrac{\pi}{2}\right\}}{\pi^{1/2}\left(\sin\dfrac{\theta}{2}\right)^{2a+1/2}\left(\cos\dfrac{\theta}{2}\right)^{2b+1/2}} n^{-1/2}+O(n^{-3/2})\right], \tag{4.15}$$

and

$$\frac{\mathrm{d}C_n(\cos\theta)}{\mathrm{d}(\cos\theta)}=2^n n!\,\frac{\Gamma(2a+2b+n+1)}{\Gamma(2a+2b+2n+1)}$$
$$\times\left[\frac{\sin\left\{\left(n+\dfrac{2a+2b+1}{2}\right)\theta-\left(2a+\dfrac{1}{2}\right)\dfrac{\pi}{2}\right\}}{2\pi^{1/2}\left(\sin\dfrac{\theta}{2}\right)^{2a+3/2}\left(\cos\dfrac{\theta}{2}\right)^{2b+3/2}}n^{1/2}+O(n^{-1/2})\right], \tag{4.16}$$

where $0<\theta<\pi$. Using eqs. (4.9)–(4.16), the correlation functions can be evaluated in the limit $N\to\infty$.

(a) global $k$-level correlation functions:

$$G_2^{(k)}(x_1,\cdots,x_k)=\mathrm{T}\det[\lim_{N\to\infty}f_2(x_i,x_j)]_k,$$

where

$$\lim_{N\to\infty}f_2(\cos\theta,\cos\theta)=\frac{N}{\pi\sin\theta},$$

and

$$\lim_{N\to\infty}f_2(\cos\theta,\cos\varphi)=S(\cos\theta,\cos\varphi)\equiv\frac{2}{\pi\sqrt{\sin\theta\sin\varphi}\,(\cos\theta-\cos\varphi)}$$
$$\times\left\{\cos\left(\nu\theta-\left(2a+\frac{1}{2}\right)\frac{\pi}{2}\right)\sin\left(\nu\varphi-\left(2a+\frac{1}{2}\right)\frac{\pi}{2}\right)\cos\frac{\theta}{2}\sin\frac{\varphi}{2}\right.$$
$$\left.-\sin\left(\nu\theta-\left(2a+\frac{1}{2}\right)\frac{\pi}{2}\right)\cos\left(\nu\varphi-\left(2a+\frac{1}{2}\right)\frac{\pi}{2}\right)\sin\frac{\theta}{2}\cos\frac{\varphi}{2}\right\}, \tag{4.17}$$
$$\nu\equiv N+a+b,\quad 0<\theta,\varphi<\pi,\quad \theta\neq\varphi.$$

The global 1-level correlation function (the level density ) is[12]

$$G_2^{(1)}(x)=\frac{N}{\pi\sqrt{1-x^2}}. \tag{4.18}$$

(b) local $k$-level correlation functinns:

$$L_2^{(k)}(w;\xi_1,\cdots,\xi_k)=\mathrm{T}\det\left[\lim_{N\to\infty}\frac{\pi\sqrt{1-w^2}}{N}f_2\left(w+\frac{\pi\sqrt{1-w^2}}{N}\xi_i,\,w+\frac{\pi\sqrt{1-w^2}}{N}\xi_j\right)\right]_k,$$

where

$$\lim_{N\to\infty}\left[\frac{\pi\sqrt{1-w^2}}{N}f_2\left(w+\frac{\pi\sqrt{1-w^2}}{N}\xi,\,w+\frac{\pi\sqrt{1-w^2}}{N}\xi\right)\right]=1,$$

and

$$\lim_{N\to\infty}\left[\frac{\pi\sqrt{1-w^2}}{N}f_2\left(w+\frac{\pi\sqrt{1-w^2}}{N}\xi,\,w+\frac{\pi\sqrt{1-w^2}}{N}\eta\right)\right]=\frac{\sin[\pi(\xi-\eta)]}{\pi(\xi-\eta)},\quad -1<w<1. \tag{4.19}$$

This formula extends Fox and Kahn's result[13] which has been derived for the special case $w=0$. We find that the local correlation functions are independent of the parameters $a$, $b$ and $w$ in the thermodynamic limit. Note that $a$ and $b$ are restricted to be finite.

(2) The Legendre ensemble

From the above result, both the global and local correlation functions of the Legendre ensemble are identical to those of the Jacobi ensemble.

(3) The Gaussian ensemble

From eqs. (2.9) and (2.10), the monic Hermite polynomials are given by

$$C_n(x)=\frac{1}{2^n}H_n(x), \tag{4.20}$$

and satisfy the orthogonality relation

$$\int e^{-2V(x)}C_m(x)C_n(x)\,dx=\int_{-\infty}^{\infty}e^{-x^2}C_m(x)C_n(x)\,dx=\frac{\pi^{1/2}n!}{2^n}\delta_{mn}=h_n\delta_{mn}. \tag{4.21}$$

The asymptotic forms of the monic Hermite polynomials are known to be[15)]

$$C_n(x)=e^{x^2/2}\frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{\sqrt{\pi}}\left[\cos\left(\sqrt{2n+1}x-\frac{n\pi}{2}\right)+O(n^{-1/2})\right], \tag{4.22}$$

and

$$C_n'(x)=-2e^{x^2/2}\frac{\Gamma\left(\frac{n}{2}+1\right)}{\sqrt{\pi}}\left[\sin\left(\sqrt{2n+1}x-\frac{n\pi}{2}\right)+O(n^{-1/2})\right], \tag{4.23}$$

where $-R<x<R$. Here and hereafter, $R$ is an arbitrary fixed positive number. Moreover we can see that

$$C_{n-1}(x)=\frac{1}{n}C_n'(x)=-2e^{x^2/2}\frac{\Gamma\left(\frac{n}{2}+1\right)}{\sqrt{\pi}}n^{-1}\left[\sin\left(\sqrt{2n+1}x-\frac{n\pi}{2}\right)+O(n^{-1/2})\right]. \tag{4.24}$$

Using eqs. (4.9)–(4.12) and (4.21)–(4.24), the correlation functions can be evaluated in the limit $N\to\infty$:

(a) global $k$-level correlation functions:

$$G_2^{(k)}(x_1,\cdots,x_k)=\mathrm{T}\det\,[\lim_{N\to\infty}f_2(x_i,x_j)]_k,$$

where

$$\lim_{N\to\infty}f_2(x,x)=\frac{\sqrt{2N}}{\pi},$$

and

$$\lim_{N\to\infty}f_2(x,y)=T(x,y)\equiv\frac{1}{\pi}\frac{\sin(\sqrt{2N+1}(x-y))}{x-y},\quad -R<x,y<R,\quad x\neq y. \tag{4.25}$$

In fact, by this definition of the global correlation functions, our observation is restricted to the vicinity of the origin because $x$ and $y$ are taken to be finite whereas the true distribution extends over the region $-\sqrt{2N}\le x,y\le\sqrt{2N}$. The "true" global level density may be evaluated by a simple physical argument.[4)] The result is

$$G_2^{(1)}(x)=\frac{1}{\pi}\sqrt{2N-x^2},\quad x^2\le 2N$$
$$=0,\quad x^2\ge 2N. \tag{4.26}$$

This is the semicircle law first derived by Wigner.[23)]

(b) local $k$-level correlation functions:

$$L_2^{(k)}(w;\xi_1,\cdots,\xi_k)=\mathrm{T}\det\left[\lim_{N\to\infty}\frac{\pi}{\sqrt{2N}}f_2\left(w+\frac{\pi}{\sqrt{2N}}\xi_i,\,w+\frac{\pi}{\sqrt{2N}}\xi_j\right)\right]_k,$$

where

$$\lim_{N\to\infty}\left[\frac{\pi}{\sqrt{2N}}f_2\left(w+\frac{\pi}{\sqrt{2N}}\xi,\,w+\frac{\pi}{\sqrt{2N}}\xi\right)\right]=1,$$

and

$$\lim_{N\to\infty}\left[\frac{\pi}{\sqrt{2N}}f_2\left(w+\frac{\pi}{\sqrt{2N}}\xi,\ w+\frac{\pi}{\sqrt{2N}}\eta\right)\right]=\frac{\sin[\pi(\xi-\eta)]}{\pi(\xi-\eta)},\quad -R<w<R. \tag{4.27}$$

We find that the local correlation functions are independent of $w$ in the thermodynamic limit. Note that $w$ is restricted to be finite.

(4) The Laguerre ensemble

From eqs. (2.11) and (2.12), the monic Laguerre polynomials are given by

$$C_n(x)=\frac{n!}{(-2)^n}L_n^{(2a)}(2x), \tag{4.28}$$

and satisfy the orthogonality relation

$$\int e^{-2V(x)}C_m(x)C_n(x)\,dx=\int_0^\infty e^{-2x}x^{2a}C_m(x)C_n(x)\,dx=\frac{n!\Gamma(2a+n+1)}{2^{2a+2n+1}}\delta_{mn}=h_n\delta_{mn}. \tag{4.29}$$

The asymptotic forms of the monic Laguerre polynomials are known to be[15]

$$C_n(x)=e^x\frac{(-1)^n n!}{\sqrt{\pi}}2^{-n-a-1/4}x^{-a-1/4}n^{a-1/4}\times\left[\cos\left(2\sqrt{(2n+2a+1)x}-a\pi-\frac{\pi}{4}\right)+O(n^{-1/2})\right], \tag{4.30}$$

and

$$C_n'(x)=e^x\frac{(-1)^{n+1}n!}{\sqrt{\pi}}2^{-n-a+1/4}x^{-a-3/4}n^{a+1/4}\times\left[\sin\left(2\sqrt{(2n+2a+1)x}-a\pi-\frac{\pi}{4}\right)+O(n^{-1/2})\right], \tag{4.31}$$

where $0<x<R$. Moreover we can see that

$$C_{n-1}(x)=-\frac{2}{n}C_n(x)+e^x\frac{(-1)^{n+1}(n-1)!}{\sqrt{\pi}}2^{-n-a+5/4}x^{-a+1/4}n^{a-3/4}\times\left[\sin\left(2\sqrt{(2n+2a+1)x}-a\pi-\frac{\pi}{4}\right)+O(n^{-1/2})\right]. \tag{4.32}$$

Using eqs. (4.9)–(4.12) and (4.29)–(4.32), the correlation functions can be evaluated in the limit $N\to\infty$:

(a) global $k$-level correlation functions:

$$G_2^{(k)}(x_1,\cdots,x_k)=\mathrm{T}\det[\lim_{N\to\infty}f_2(x_i,x_j)]_k,$$

where

$$\lim_{N\to\infty}f_2(x,x)=\frac{1}{\pi}\sqrt{\frac{2N}{x}},$$

and

$$\lim_{N\to\infty}f_2(x,y)=\frac{1}{\pi(x-y)x^{1/4}y^{1/4}}\left\{\sqrt{x}\sin\left(2\sqrt{\gamma x}-a\pi-\frac{\pi}{4}\right)\cos\left(2\sqrt{\gamma y}-a\pi-\frac{\pi}{4}\right)-\sqrt{y}\cos\left(2\sqrt{\gamma x}-a\pi-\frac{\pi}{4}\right)\sin\left(2\sqrt{\gamma y}-a\pi-\frac{\pi}{4}\right)\right\},$$
$$\gamma\equiv 2N+2a+1,\quad 0<x,y<R,\quad x\neq y. \tag{4.33}$$

Similarly to the case of the Gaussian ensemble, we are restricted to the vicinity of the origin because $x$ and $y$ are taken to be finite. The "true" global level density may be evaluated by a simple physical argument.[14] The result is

$$G_2^{(1)}(x)=\frac{1}{\pi\sqrt{x}}\sqrt{2N-x}, \quad 0<x\leq 2N, \qquad =0, \quad x\geq 2N. \tag{4.34}$$

(b) local $k$-level correlation functions:

$$L_2^{(k)}(w;\xi_1,\cdots,\xi_k)=\mathrm{T}\det\left[\lim_{N\to\infty}\pi\sqrt{\frac{w}{2N}}f_2\left(w+\pi\sqrt{\frac{w}{2N}}\xi_i,\ w+\pi\sqrt{\frac{w}{2N}}\xi_j\right)\right]_k,$$

where

$$\lim_{N\to\infty}\left[\pi\sqrt{\frac{w}{2N}}f_2\left(w+\pi\sqrt{\frac{w}{2N}}\xi,\ w+\pi\sqrt{\frac{w}{2N}}\xi\right)\right]=1,$$

and

$$\lim_{N\to\infty}\left[\pi\sqrt{\frac{w}{2N}}f_2\left(w+\pi\sqrt{\frac{w}{2N}}\xi,\ w+\pi\sqrt{\frac{w}{2N}}\eta\right)\right]=\frac{\sin[\pi(\xi-\eta)]}{\pi(\xi-\eta)}, \quad 0<w<R. \tag{4.35}$$

This is the explicit formula for the result reported by Fox and Kahn.[13] We find that the local correlation functions are independent of $a$ and $w$ in the thermodynamic limit. Note that $a$ and $w$ are restricted to be finite.

As far as we know, the local correlation functions of complex hermitian random matrix ensembles related to classical orthogonal polynomials are all identical.

## §5. Real Symmetric Random Matrices

In this section we investigate physically the most interesting case $\beta=1$. This case is mathematically more involved than the case $\beta=2$, because we must deal with skew orthogonal polynomials.[9,20,24-27)]

We define monic skew orthogonal polynomials of the second kind

$$R_n(x)=x^n+\cdots, \tag{5.1}$$

which satisfy

$$\begin{aligned}&\langle R_{2m}(x), R_{2n+1}(y)\rangle_R=-\langle R_{2n+1}(x), R_{2m}(y)\rangle_R=r_m\delta_{mn},\\&\langle R_{2m}(x), R_{2n}(y)\rangle_R=0,\\&\langle R_{2m+1}(x), R_{2n+1}(y)\rangle_R=0,\end{aligned} \tag{5.2}$$

where

$$\langle f(x), g(y)\rangle_R\equiv\int e^{-V(x)}e^{-V(y)}f(x)g(y)\varepsilon(y-x)\,dx\,dy,$$

with

$$\varepsilon(x)=\frac{1}{2}\frac{x}{|x|}.$$

We consider the $k$-level correlation functions for $N=2\nu$ particles. In the following, we assume $k\leq\nu$. Then we have[26]

$$I_{2\nu 1}^{(k)}(x_1,\cdots x_k)=\frac{1}{(2\nu-k)!}\int\prod_{i=1}^{2\nu}\mathrm{e}^{-V(x_i)}\prod_{i<j}^{2\nu}|x_i-x_j|\,\mathrm{d}x_{k+1}\mathrm{d}x_{k+2}\cdots\mathrm{d}x_{2\nu}$$

$$=\frac{(2\nu)!}{\nu!(2\nu-k)!}\int\begin{vmatrix}\vartheta_0(x_1) & \vartheta_1(x_1) & \vdots & \vartheta_{2\nu-1}(x_1)\\ \vartheta_0'(x_1) & \vartheta_1'(x_1) & \cdots & \vartheta_{2\nu-1}'(x_1)\\ \vdots & \vdots & \vdots & \vdots\\ \vartheta_0(x_\nu) & \vartheta_1(x_\nu) & \cdots & \vartheta_{2\nu-1}(x_\nu)\\ \vartheta_0'(x_\nu) & \vartheta_1'(x_\nu) & \cdots & \vartheta_{2\nu-1}'(x_\nu)\end{vmatrix}\mathrm{d}x_{k+1}\cdots\mathrm{d}x_\nu$$

$$=\frac{(2\nu)!2^\nu}{\nu!(2\nu-k)!}\left(\prod_{m=0}^{\nu-1}r_m\right)\int\det C(M)\,\mathrm{d}x_{k+1}\cdots\mathrm{d}x_\nu, \tag{5.3}$$

where $M$ is a $\nu\times\nu$ quaternion matrix whose elements are

$$C(M_{ij})=\frac{1}{\sqrt{2r_{j-1}}}\begin{pmatrix}\vartheta_{2j-2}(x_i) & \vartheta_{2j-1}(x_i)\\ \vartheta_{2j-2}'(x_i) & \vartheta_{2j-1}'(x_i)\end{pmatrix}, \tag{5.4}$$

with

$$\vartheta_i(x)\equiv\int_{-\infty}^{x}\mathrm{d}y\,\mathrm{e}^{-V(y)}R_i(y). \tag{5.5}$$

Since a quaternion matrix $M\bar{M}$ is self-dual, we can apply *Theorem 1* and get

$$\det C(M)=\det C(\bar{M})=[\det C(M\bar{M})]^{1/2}=\mathrm{T}\det(M\bar{M}). \tag{5.6}$$

We define

$$f_1(x_i,x_j)\equiv(M\bar{M})_{ij}=\begin{bmatrix}S_1(x_i,x_j) & I_1(x_i,x_j)\\ D_1(x_i,x_j) & S_1(x_j,x_i)\end{bmatrix}, \tag{5.7}$$

where

$$S_1(x,y)=\sum_{m=0}^{\nu-1}\frac{1}{2r_m}\{\vartheta_{2m}(x)\vartheta_{2m+1}'(y)-\vartheta_{2m+1}(x)\vartheta_{2m}'(y)\},$$

$$I_1(x,y)=-\int_{-\infty}^{y}S_1(x,y')\,\mathrm{d}y',\quad D_1(x,y)=\frac{\partial}{\partial x}S_1(x,y).$$

It is straightforward to see that $f_1(x,y)$ satisfies

$$\bar{f}_1(y,x)=f_1(x,y), \tag{5.8}$$

$$\int f_1(x,x)\,\mathrm{d}x=\nu, \tag{5.9}$$

and

$$\int f_1(x,y)f_1(y,z)\,\mathrm{d}y=f_1(x,z). \tag{5.10}$$

Therefore we can apply *Theorem 2* to the evaluation of (5.3). The $k$-level functions and the partition function, respectively, are expressed by

$$I_{2\nu 1}^{(k)}(x_1,x_2,\cdots,x_k)=\frac{(2\nu)!2^\nu}{\nu!(2\nu-k)!}\left(\prod_{m=0}^{\nu-1}r_m\right)\int\mathrm{T}\det[f_1(x_i,x_j)]_N\,\mathrm{d}x_{k+1}\cdots\mathrm{d}x_\nu$$

$$=\frac{(2\nu)!(\nu-k)!2^\nu}{\nu!(2\nu-k)!}\left(\prod_{m=0}^{\nu-1}r_m\right)\mathrm{T}\det[f_1(x_i,x_j)]_k, \tag{5.11}$$

$$I_{2\nu 1}^{(0)}=2^{\nu}\prod_{m=0}^{\nu-1} r_m. \tag{5.12}$$

Thus we see again that if $r_m$'s and $f_1(x, y)$ are given, all the thermodynamic quantities are determined.

Assuming that $V(x)=\infty$ at the boundaries, we may prove that

$$\langle V'(x)f(x), g(y)\rangle_R-\langle f'(x), g(y)\rangle_R=-\int e^{-2V(x)}f(x)g(x)\,dx. \tag{5.13}$$

This formula will be used for the evaluation of skew orthogonal polynomials.

(1) The Jacobi ensemble

We now present the monic skew Jacobi polynomials of the second kind and their normalization constants $r_n$'s. In order to make the formula (5.13) applicable, we shall restrict ourselves to the case $a>0$, $b>0$. From eqs. (2.13) and (5.12), we find that

$$r_n=\frac{2^{2a+2b+4n+4}(2n)!\,\Gamma(2a+2n+2)\Gamma(2b+2n+2)\Gamma(2a+2b+2n+3)}{\Gamma(2a+2b+4n+3)\Gamma(2a+2b+4n+5)}. \tag{5.14}$$

The monic skew Jacobi polynomials $R_n(x)$ can be expanded by using the monic Jacobi polynomials $C_n(x)$ as follows:

$$\begin{aligned}R_{2m}(x)&=a_{m0}C_{2m}(x)+a_{m1}C_{2m-1}(x)+a_{m2}C_{2m-2}(x)+\cdots,\\ R_{2m+1}(x)&=b_{m0}C_{2m+1}(x)+b_{m1}C_{2m}(x)+b_{m2}C_{2m-1}(x)+\cdots,\end{aligned} \tag{5.15}$$

where

$$a_{m0}=b_{m0}=1,$$

and

$$C_n(x)=2^n n!\,\frac{\Gamma(2a+2b+n+1)}{\Gamma(2a+2b+2n+1)}P_n^{(2a,2b)}(x). \tag{5.16}$$

We evaluate the coefficients $a_{mn}$'s and $b_{mn}$'s. As for the monic Jacobi polynomials, the following property is known.

$$D_{n+1}(x)\equiv(1-x^2)C_n'(x)=t_nC_{n+1}(x)+u_nC_n(x)+v_nC_{n-1}(x), \tag{5.17}$$

where

$$t_n=-n,\quad u_n=(a-b)\frac{n(2a+2b+n+1)}{(a+b+n)(a+b+n+1)},$$

and

$$v_n=\frac{n(2a+n)(2b+n)(2a+2b+n)(2a+2b+n+1)}{(a+b+n)^2(2a+2b+2n-1)(2a+2b+2n+1)}.$$

Substituting $D_{2m-n}(x)$, $n\geq 0$ and $R_{2m}(x)$ for $f(x)$ and $g(x)$ in eq. (5.13), and noting that $R_{2m}(x)$ is skew orthogonal to any polynomial of degree less than or equal to $2m$, we get a three-term recurrence relation for $a_{mn}$'s:

$$T_{mn}a_{mn}+U_{mn}a_{mn+1}+V_{mn}a_{mn+2}=0,\quad n\geq 0. \tag{5.18}$$

Substituting $D_{2m-n+1}(x)$, $n\geq 1$ and $R_{2m+1}(x)$ for $f(x)$ and $g(x)$ in eq. (5.13), and noting that $R_{2m+1}(x)$ is skew orthogonal to any polynomial of degree less than $2m$, we get a three-term recurrence relation for $b_{mn}$'s:

$$T_{mn-1}b_{mn}+U_{mn-1}b_{mn+1}+V_{mn-1}b_{mn+2}=0,\quad n\geq 1. \tag{5.19}$$

Here $T'_{mn}$'s, $U_{mn}$'s and $V_{mn}$'s are given by

$$T_{mn}=-(2m-n)(2m-n-1)(2a+2m-n)(2b+2m-n)(a+b+2m-n-1),$$

$$U_{mn}=(a-b)(2m-n-1)(a+b+2m-n)(2a+2b+4m-2n-1)(2a+2b+4m-2n+1),$$

and

$$V_{mn}=(a+b+2m-n)^2(2a+2b+4m-2n-1)(2a+2b+4m-2n+1)(a+b+2m-n-1).$$

By similar substitutions, we get

$$a_{m1}=-\frac{2m(a-b)}{(a+b+2m)(a+b+2m+1)}, \tag{5.20}$$

$$b_{m1}=c_m,$$

$$b_{m2}=-\frac{2m(a-b)c_m}{(a+b+2m)(a+b+2m+1)} -\frac{2(a+b)m(2a+2m+1)(2b+2m+1)}{(a+b+2m+1)^2(a+b+2m+2)(2a+2b+4m+1)}. \tag{5.21}$$

Here and hereafter, $c_m$ denotes an arbitrary constant. It has contribution to neither the skew orthogonal relations nor the correlation functions. Now we take $c_m$ to be zero. Then we get the following expressions for $a_{mn}$'s and $b_{mn}$'s. In the case $a\neq b$, the $a_{mn}$'s are given by

$$a_{mn}=(-1)^n a_{m0}\prod_{l=1}^{n}\chi_{ml}, \quad n\geq 1, \tag{5.22}$$

where

$$\chi_{ml}\equiv\alpha_{ml}-\frac{\beta_{ml}|}{|\alpha_{ml-1}}-\frac{\beta_{ml-1}|}{|\alpha_{ml-2}}-\cdots-\frac{\beta_{m2}|}{|\alpha_{m1}},$$

with

$$\alpha_{mn}\equiv\frac{U_{mn-2}}{V_{mn-2}}, \quad \beta_{mn}\equiv\frac{T_{mn-2}}{V_{mn-2}}, \quad n\geq 2,$$

$$\chi_{m1}=\alpha_{m1}\equiv-\frac{a_{m1}}{a_{m0}}.$$

The $b_{mn}$'s are given by

$$b_{mn}=(-1)^n b_{m2}\prod_{l=3}^{n}\chi'_{ml}, \quad n\geq 3, \tag{5.23}$$

where

$$\chi'_{ml}\equiv\alpha'_{ml}-\frac{\beta'_{ml}|}{|\alpha'_{ml-1}}-\frac{\beta'_{ml-1}|}{|\alpha'_{ml-2}}-\cdots-\frac{\beta'_{m4}|}{|\alpha'_{m3}},$$

with

$$\alpha'_{mn}\equiv\frac{U_{mn-3}}{V_{mn-3}}, \quad \beta'_{mn}\equiv\frac{T_{mn-3}}{V_{mn-3}}, \quad n\geq 4,$$

$$\chi'_{m3}=\alpha'_{m3}\equiv-\frac{b_{m3}}{b_{m2}}.$$

Here we have used the notation for continued fractions

$$x_1+\frac{x_2|}{|x_3}+\frac{x_4|}{|x_5}+\cdots\equiv x_1+\cfrac{x_2}{x_3+\cfrac{x_4}{x_5+\cdots}}. \tag{5.24}$$

In the case $a=b$, eqs. (5.18) and (5.19) become two-term recurrence relations and the expressions for monic skew orthogonal polynomials are simplified as

$$R_{2m}(x)=C_{2m}(x)+\sum_{n=1}^{m}\left\{\prod_{l=0}^{n-1}\frac{(2m-2l)(2m-2l-1)}{(4a+4m-4l-1)(4a+4m-4l+1)}\right\}C_{2m-2n}(x), \tag{5.25}$$

$$R_{2m+1}(x)=C_{2m+1}(x)+b_{m2}C_{2m-1}(x) +b_{m2}\sum_{n=2}^{m}\left\{\prod_{l=1}^{n-1}\frac{(2m-2l+1)(2m-2l)}{(4a+4m-4l+1)(4a+4m-4l+3)}\right\}C_{2m-2n+1}(x), \tag{5.26}$$

where

$$b_{m2}=\frac{-2ma}{(a+m+1)(4a+4m+1)},$$

and

$$C_n(x)=2^n n!\,\frac{\Gamma(4a+n+1)}{\Gamma(4a+2n+1)}P_n^{(2a,2a)}(x). \tag{5.27}$$

In terms of the incomplete Beta function

$$B(x;\lambda,\mu)\equiv\int_{-1}^{x}(1-t)^{\lambda}(1+t)^{\mu}\,\mathrm{d}t,\quad -1<x<1,\ \lambda,\mu\geq 0, \tag{5.28}$$

we can evaluate the $k$-level correlation functions for arbitrary $N$ and $k\leq(N/2)$ using the above results.

(2) The Legendre ensemble

Now we consider a limiting case, $a=b=0$. Monic skew orthogonal polynomials are drastically symplified into

$$R_{2m}(x)=\frac{1}{2m+1}C'_{2m+1}(x), \tag{5.29}$$

$$R_{2m+1}(x)=C_{2m+1}(x), \tag{5.30}$$

where

$$C_n(x)=2^n n!\,\frac{\Gamma(n+1)}{\Gamma(2n+1)}P_n^{(0,0)}(x). \tag{5.31}$$

Using the above formulae, we can evaluate the global and local correlation functions. The results are as follows.[8,9)]

(a) global $k$-level correlation functions:

$$G_1^{(k)}(x_1,\cdots,x_k)=2^k\,\mathrm{T\,det}\,[\lim_{N\to\infty}f_1(x_i,x_j)]_k, \tag{5.32}$$

where

$$\lim_{N\to\infty}S_1(\cos\theta,\cos\theta)=\frac{N}{2\pi\sin\theta},$$

and

$$\lim_{N\to\infty}S_1(\cos\theta,\cos\varphi)=\frac{1}{4}\sqrt{\frac{2N}{\pi}}\frac{1}{(\sin\varphi)^{3/2}}\sin\left\{\left(N+\frac{1}{2}\right)\varphi-\frac{\pi}{4}\right\}, \tag{5.33}$$

with

$$0<\theta,\quad \varphi<\pi,\quad \theta\neq\varphi.$$

The global 1-level correlation function (the level density) is

$$G_1^{(1)}(x)=\frac{N}{\pi\sqrt{1-x^2}}. \tag{5.34}$$

This is identical to the level density formula (4.18) for $\beta=2$.

(b) local $k$-level correlation functions:

$$L_1^{(k)}(w;\xi_1,\cdots,\xi_k)=2^k\,\mathrm{T\,det}\left[\lim_{N\to\infty}\frac{\pi\sqrt{1-w^2}}{N}f_1\left(w+\frac{\pi\sqrt{1-w^2}}{N}\xi_i,\ w+\frac{\pi\sqrt{1-w^2}}{N}\xi_j\right)\right]_k, \tag{5.35}$$

where

$$\lim_{N\to\infty}\frac{\pi\sqrt{1-w^2}}{N}S_1\left(w+\frac{\pi\sqrt{1-w^2}}{N}\xi,\ w+\frac{\pi\sqrt{1-w^2}}{N}\xi\right)=\frac{1}{2},$$

$$\lim_{N\to\infty}\frac{\pi\sqrt{1-w^2}}{N}S_1\left(w+\frac{\pi\sqrt{1-w^2}}{N}\xi,\ w+\frac{\pi\sqrt{1-w^2}}{N}\eta\right)=\frac{\sin[\pi(\xi-\eta)]}{2\pi(\xi-\eta)},\quad -1<w<1.$$

(3) The Gaussian ensemble

We shall deal with the famous Gaussian ensemble at $\beta=1$, the Gaussian orthogonal ensemble. Firstly, we derive the monic skew Hermite polynomials of the second kind and their normalization constants $r_n$'s. From eqs. (2.15) and (5.12), we find that

$$r_n=2^{-2n}\sqrt{\pi}\,\Gamma(2n+1). \tag{5.36}$$

We expand the monic skew Hermite polynomials $R_n(x)$ by the monic Hermite polynomials $C_n(x)$ as follows:

$$R_{2m}(x)=a_{m0}C_{2m}(x)+a_{m2}C_{2m-2}(x)+a_{m4}C_{2m-4}(x)+\cdots,$$
$$R_{2m+1}(x)=b_{m0}C_{2m+1}(x)+b_{m2}C_{2m-1}(x)+b_{m4}C_{2m-3}(x)+\cdots, \tag{5.37}$$

where

$$a_{m0}=b_{m0}=1,$$

and

$$C_n(x)=\frac{1}{2^n}H_n(x). \tag{5.38}$$

We evaluate the coefficients $a_{mn}$'s and $b_{mn}$'s. Substituting $C_{2m-n}(x)$, $n\geq 2$ and $R_{2m}(x)$ for $f(x)$ and $g(x)$ in eq. (5.13), we get

$$a_{mn}=0,\quad n\geq 2. \tag{5.39}$$

Substituting $C_{2m-n+1}(x)$, $n\geq 4$, and $R_{2m+1}(x)$ for $f(x)$ and $g(x)$ in eq. (5.13), we get

$$b_{mn}=0,\quad n\geq 4. \tag{5.40}$$

Substituting $C_{2m-1}(x)$ and $R_{2m+1}(x)$ for $f(x)$ and $g(x)$ in eq. (5.13), we get

$$b_{m2}=-m. \tag{5.41}$$

Hence we find that the monic skew Hermite polynomials of the second kind are given by

$$R_{2m}(x)=C_{2m}(x),$$
$$R_{2m+1}(x)=C_{2m+1}(x)-mC_{2m-1}(x). \tag{5.42}$$

Using the above formulae, we can evaluate the global and local correlation functions. The results are as follows.

(a) global $k$-level correlation functions:

$$G_1^{(k)}(x_1,\cdots,x_k)=2^k\,\mathrm{T}\det[\lim_{N\to\infty}f_1(x_i,x_j)]_k, \tag{5.43}$$

where

$$\lim_{N\to\infty}S_1(x,x)=\sqrt{\frac{N}{2}}\frac{1}{\pi}, \tag{5.44}$$

with

$$-R<x<R.$$

In this case we are again restricted to the vicinity of the origin and the global 1-level correlation function (the level density) is evaluated by the same physical argument as before. The result is

$$G_1^{(1)}(x)=\frac{1}{\pi}\sqrt{2N-x^2},\quad x^2\leq 2N$$
$$=0,\quad x^2\geq 2N. \tag{5.45}$$

This is identical to the level density formula (4.26) for $\beta=2$.

(b) local $k$-level correlation functions:

$$L_1^{(k)}(w;\xi_1,\cdots,\xi_k)=2^k\,\mathrm{T}\det\left[\lim_{N\to\infty}\frac{\pi}{\sqrt{2N}}f_1\left(w+\frac{\pi}{\sqrt{2N}}\xi_i,\ w+\frac{\pi}{\sqrt{2N}}\xi_j\right)\right]_k, \tag{5.46}$$

where

$$\lim_{N\to\infty}\frac{\pi}{\sqrt{2N}}S_1\left(w+\frac{\pi}{\sqrt{2N}}\xi,\, w+\frac{\pi}{\sqrt{2N}}\xi\right)=\frac{1}{2},$$
$$\lim_{N\to\infty}\frac{\pi}{\sqrt{2N}}S_1\left(w+\frac{\pi}{\sqrt{2N}}\xi,\, w+\frac{\pi}{\sqrt{2N}}\eta\right)=\frac{\sin[\pi(\xi-\eta)]}{2\pi(\xi-\eta)},\quad -R<w<R.$$

(4) The Laguerre ensemble
We restrict ourselves to the case $a>0$. From eqs. (2.16) and (5.12), we find that

$$r_n=2^{-2a-4n-2}\Gamma(2n+1)\Gamma(2a+2n+2). \tag{5.47}$$

The monic skew Laguerre polynomials $R_n(x)$ can be expanded by the monic Laguerre polynomials $C_n(x)$ as follows:

$$R_{2m}(x)=a_{m0}C_{2m}(x)+a_{m1}C_{2m-1}(x)+a_{m2}C_{2m-2}(x)+\cdots,$$
$$R_{2m+1}(x)=b_{m0}C_{2m+1}(x)+b_{m1}C_{2m}(x)+b_{m2}C_{2m-1}(x)+\cdots, \tag{5.48}$$

where

$$a_{m0}=b_{m0}=1,$$

and

$$C_n(x)=\frac{n!}{(-2)^n}L_n^{(2a)}(2x). \tag{5.49}$$

We evaluate the coefficients $a_{mn}$'s and $b_{mn}$'s. As for the monic Laguerre polynomials, the following property is known.

$$D_n(x)\equiv xC_n'(x)$$
$$=nC_n(x)+\frac{n(2a+n)}{2}C_{n-1}(x). \tag{5.50}$$

Substituting $D_{2m-n}(x)$, $n\geq 0$ and $R_{2m}(x)$ for $f(x)$ and $g(x)$ in eq. (5.13), we get

$$(2m-n)a_{mn}+2a_{mn+1}=0,\quad n\geq 0. \tag{5.51}$$

Substituting $D_{2m-n+1}(x)$, $n\geq 2$, and $R_{2m+1}(x)$ for $f(x)$ and $g(x)$ in eq. (5.13), we get

$$(2m-n+1)b_{mn}+2b_{mn+1}=0,\quad n\geq 2. \tag{5.52}$$

By similar substitutions, we get

$$b_{m1}=c_m,\ b_{m2}=-\frac{m}{2}(2c_m+2a+2m+1). \tag{5.53}$$

Thus the $a_{mn}$'s are given by

$$a_{mn}=\frac{1}{(-2)^n}\prod_{l=1}^{n}(2m-l+1),\quad n\geq 1, \tag{5.54}$$

and $b_{mn}$'s are given by

$$b_{mn}=\frac{b_{m2}}{(-2)^{n-2}}\prod_{l=1}^{n-2}(2m-l),\quad n\geq 3. \tag{5.55}$$

In terms of the incomplete Gamma function

$$\Gamma(x;\lambda)\equiv\int_0^x e^{-t}t^{\lambda}\,dt,\quad x>0,\quad \lambda\geq 0, \tag{5.56}$$

we can evaluate the $k$-level correlation functions for arbitrary $N$ and $k\leq(N/2)$ using the above results.

## §6. Self-Dual Quaternion Random Matrices

In the case of self-dual quaternion random matrices, we need another kind of skew orthogonal polynomials. We define monic skew orthogonal polynomials of the first kind

$$Q_n(x)=x^n+\cdots, \tag{6.1}$$

which satisfy

$$\langle Q_{2m}(x),Q_{2n+1}(y)\rangle_Q=-\langle Q_{2n+1}(x),Q_{2m}(y)\rangle_Q=q_m\delta_{mn},$$
$$\langle Q_{2m}(x),Q_{2n}(y)\rangle_Q=0,$$
$$\langle Q_{2m+1}(x),Q_{2n+1}(y)\rangle_Q=0, \tag{6.2}$$

where

$$\langle f(x), g(y)\rangle_Q \equiv \frac{1}{2}\int e^{-4V(x)}[f(x)g'(x)-f'(x)g(x)]\,dx.$$

Then we can express the $k$-level correlation functions for $N$ particles as[26]

$$I_{N4}^{(k)}(x_1, \cdots, x_k)=\frac{1}{(N-k)!}\int \prod_{i=1}^{N} e^{-4V(x_i)} \prod_{i<j}^{N}(x_i-x_j)^4\,dx_{k+1}\,dx_{k+2}\cdots dx_N$$

$$=\frac{1}{(N-k)!}\int \prod_{i=1}^{N} e^{-4V(x_i)} \begin{vmatrix} Q_0(x_1) & Q_1(x_1) & \cdots & Q_{2N-1}(x_1) \\ Q_0'(x_1) & Q_1'(x_1) & \cdots & Q_{2N-1}'(x_1) \\ \vdots & \vdots & \vdots & \vdots \\ Q_0(x_N) & Q_1(x_N) & \cdots & Q_{2N-1}(x_N) \\ Q_0'(x_N) & Q_1'(x_N) & \cdots & Q_{2N-1}'(x_N) \end{vmatrix} dx_{k+1}\cdots dx_N$$

$$=\frac{2^N}{(N-k)!}\left(\prod_{m=0}^{N-1} q_m\right)\int \det C(M)\,dx_{k+1}\cdots dx_N, \tag{6.3}$$

where $M$ is $N\times N$ quaternion matrix whose elements are

$$C(M_{ij})=\frac{e^{-2V(x_i)}}{\sqrt{2q_{j-1}}}\begin{pmatrix} Q_{2j-2}(x_i) & Q_{2j-1}(x_i) \\ Q_{2j-2}'(x_i) & Q_{2j-1}'(x_i)\end{pmatrix}. \tag{6.4}$$

A quaternion matrix $M\bar{M}$ is self-dual, and therefore we can apply *Theorem 1* to get

$$\det C(M)=\det C(\bar{M})=[\det C(M\bar{M})]^{1/2}=\mathrm{T}\det(M\bar{M}). \tag{6.5}$$

We define

$$f_4(x_i, x_j)\equiv(M\bar{M})_{ij}=\begin{bmatrix} S_4(x_i, x_j) & I_4(x_i, x_j) \\ D_4(x_i, x_j) & S_4(x_j, x_i)\end{bmatrix}, \tag{6.6}$$

where

$$S_4(x, y)=\sum_{m=0}^{N-1}\frac{e^{-2[V(x)+V(y)]}}{2q_m}\{Q_{2m}(x)Q_{2m+1}'(y)-Q_{2m+1}(x)Q_{2m}'(y)\},$$

$$I_4(x, y)=-\sum_{m=0}^{N-1}\frac{e^{-2[V(x)+V(y)]}}{2q_m}\{Q_{2m}(x)Q_{2m+1}(y)-Q_{2m+1}(x)Q_{2m}(y)\},$$

and

$$D_4(x, y)=\sum_{m=0}^{N-1}\frac{e^{-2[V(x)+V(y)]}}{2q_m}\{Q_{2m}'(x)Q_{2m+1}'(y)-Q_{2m+1}'(x)Q_{2m}'(y)\}.$$

It is straightforward to see that $f_4(x, y)$ satisfies

$$\bar{f}_4(y, x)=f_4(x, y), \tag{6.7}$$

$$\int f_4(x, x)\,dx=N, \tag{6.8}$$

and

$$\int f_4(x, y)f_4(y, z)\,dy=f_4(x, z). \tag{6.9}$$

Therefore we can use *Theorem 2* in the evaluation of (6.3). The $k$-level correlation functions and the partition function, respectively, are expressed by

$$I_{N4}^{(k)}(x_1, x_2, \cdots, x_k)=\frac{2^N}{(N-k)!}\left(\prod_{m=0}^{N-1} q_m\right)\int \mathrm{T}\det[f_4(x_i, x_j)]_N \, dx_{k+1}\cdots dx_N$$
$$=2^N\left(\prod_{m=0}^{N-1} q_m\right)\mathrm{T}\det[f_4(x_i, x_j)]_k, \tag{6.10}$$

$$I_{N4}^{(0)}=2^N\prod_{m=0}^{N-1} q_m. \tag{6.11}$$

We again see that if $q_m$'s and $f_4(x, y)$ are given, all the thermodynamic quantities are determined.

Assuming that $V(x)=\infty$ at the boundaries, we readily prove that

$$\langle f(x), g(y)\rangle_Q=2\int e^{-4V(x)}V'(x)f(x)g(x)\,dx-\int e^{-4V(x)}f'(x)g(x)\,dx. \tag{6.12}$$

This formula will be useful for the evaluation of skew orthogonal polynomials.

(1) The Jacobi ensemble

We present the monic skew Jacobi polynomials of the first kind and their normalization constants $q_n$'s. We shall restrict ourselves to the case $a>0$, $b>0$. From eqs. (2.13) and (6.11), we find that

$$q_n=\frac{2^{4a+4b+4n-1}\Gamma(2n+3)\Gamma(4a+2n+1)\Gamma(4b+2n+1)\Gamma(4a+4b+2n)}{(n+1)\Gamma(4a+4b+4n)\Gamma(4a+4b+4n+2)}. \tag{6.13}$$

Using the monic Jacobi polynomials $C_n(x)$, the monic skew Jacobi polynomials $Q_n(x)$ are expanded as

$$Q_{2m}(x)=a_{m0}C_{2m}(x)+a_{m1}C_{2m-1}(x)+a_{m2}C_{2m-2}(x)+\cdots,$$
$$Q_{2m+1}(x)=b_{m0}C_{2m+1}(x)+b_{m1}C_{2m}(x)+b_{m2}C_{2m-1}(x)+\cdots, \tag{6.14}$$

where

$$a_{m0}=b_{m0}=1,$$

and

$$C_n(x)=2^n n!\frac{\Gamma(4a+4b+n+1)}{\Gamma(4a+4b+2n+1)}P_n^{(4a,4b)}(x). \tag{6.15}$$

We evaluate the coefficients $a_{mn}$'s and $b_{mn}$'s. As for the monic Jacobi polynomials, the following property is known.

$$(1-x^2)C'_n(x)=t_nC_{n+1}(x)+u_nC_n(x)+v_nC_{n-1}(x), \tag{6.16}$$

where

$$t_n=-n,\ u_n=(a-b)\frac{2n(4a+4b+n+1)}{(2a+2b+n)(2a+2b+n+1)},$$

and

$$v_n=\frac{n(4a+n)(4b+n)(4a+4b+n)(4a+4b+n+1)}{(2a+2b+n)^2(4a+4b+2n-1)(4a+4b+2n+1)}.$$

In addition, the monic Jacobi polynomials obey the following recurrence relations:

$$xC_n(x)=C_{n+1}(x)+\sigma_nC_n(x)+\tau_nC_{n-1}(x), \tag{6.17}$$

where

$$\sigma_n=-\frac{4(a^2-b^2)}{(2a+2b+n)(2a+2b+n+1)},$$

and

$$\tau_n=\frac{n(4a+n)(4b+n)(4a+4b+n)}{(2a+2b+n)^2(4a+4b+2n-1)(4a+4b+2n+1)}.$$

We substitute $(1-x^2)C_{2m-n-1}(x)$, $n\geq 1$, and $Q_{2m}(x)$ for $f(x)$ and $g(x)$ in eq. (6.12). Noting that $Q_{2m}(x)$ is skew orthogonal to any polynomial of degree less than or equal to $2m$ and using eqs. (6.16) and (6.17), we get a three-term recurrence relation for $a_{mn}$'s:

$$T_{mn}a_{mn}+U_{mn}a_{mn+1}+V_{mn}a_{mn+2}=0, \quad n\geq 1. \tag{6.18}$$

We substitute $(1-x^2)C_{2m-n}(x)$, $n\geq 3$, and $Q_{2m+1}(x)$ for $f(x)$ and $g(x)$ in eq. (6.12). Noting that $Q_{2m+1}(x)$ is skew orthogonal to any polynomial of degree less than $2m$ and using eqs. (6.16) and (6.17), we get a three-term recurrence relation for $b_{mn}$'s:

$$T_{mn-1}b_{mn}+U_{mn-1}b_{mn+1}+V_{mn-1}b_{mn+2}=0, \quad n\geq 3. \tag{6.19}$$

Here $T_{mn}$'s, $U_{mn}$'s and $V_{mn}$'s are given by

$$T_{mn}=(2m-n)(4a+2m-n)(4b+2m-n)(4a+4b+2m-n)\times(2a+2b+2m-n-1)(2a+2b+2m-n+1), \tag{6.20}$$

$$U_{mn}=-4(a^2-b^2)(2a+2b+2m-n)(4a+4b+4m-2n-1)(4a+4b+4m-2n+1), \tag{6.21}$$

and

$$V_{mn}=-(2a+2b+2m-n-2)(2a+2b+2m-n-1)(2a+2b+2m-n)^2\times(4a+4b+4m-2n-1)(4a+4b+4m-2n+1). \tag{6.22}$$

By similar substitutions, we get

$$a_{m1}=\frac{2m(a-b)}{(a+b+m)(2a+2b+2m-1)},$$
$$a_{m2}=\frac{2m(a+b)(4a+2m-1)(4b+2m-1)}{(a+b+m-1)(2a+2b+2m-1)^2(4a+4b+4m-1)}, \tag{6.23}$$

$$b_{m1}=c_m,$$
$$b_{m2}=\frac{2m(a-b)c_m}{(a+b+m)(2a+2b+2m-1)}-\frac{2m(2m+1)}{(2a+2b+2m-1)(2a+2b+2m+1)}\frac{(16a^2+16b^2-16ab+8am+8bm+4m^2-1)}{(4a+4b+4m-1)(4a+4b+4m+1)},$$
$$b_{m3}=\frac{2m(a+b)}{(2a+2b+2m-1)^2(2a+2b+2m+1)}\frac{(4a+2m-1)(4b+2m-1)}{(a+b+m-1)(a+b+m)(4a+4b+4m-1)}\times[(a+b+m)(2a+2b+2m+1)c_m-(a-b)(2m+1)],$$
$$b_{m4}=\frac{4m(m-1)(a-b)(2a+2b+m-1)}{(2a+2b+2m-3)(2a+2b+2m-1)(2a+2b+2m+1)}\times\frac{(4a+2m-1)(4b+2m-1)}{(a+b+m-1)^2(a+b+m)(4a+4b+4m-3)(4a+4b+4m-1)}\times[(a+b+m)(2a+2b+2m+1)c_m-(a-b)(2m+1)]. \tag{6.24}$$

Consequently, $a_{mn}$'s are given by

$$a_{m2n}=a_{m2}\prod_{l=1}^{n-1}\left[\frac{4(m-l)(4a+2m-2l-1)(4b+2m-2l-1)}{(2a+2b+2m-2l-1)^2(2a+2b+2m-2l-2)}\right.$$
$$\left.\times\frac{(2a+2b+m-l)(2a+2b+2m-2l+1)}{(4a+4b+4m-4l-1)(4a+4b+4m-4l+1)}\right],\quad n\geq 2, \tag{6.25}$$

$$a_{m2n+1}=a_{m1}\prod_{l=1}^{n}\left[\frac{4(m-l)(4a+2m-2l+1)(4b+2m-2l+1)}{(2a+2b+2m-2l)^2(2a+2b+2m-2l-1)}\right.$$
$$\left.\times\frac{(2a+2b+m-l)(2a+2b+2m-2l+2)}{(4a+4b+4m-4l+1)(4a+4b+4m-4l+3)}\right],\quad n\geq 1, \tag{6.26}$$

and $b_{mn}$'s are given by

$$b_{m2n}=b_{m4}\prod_{l=1}^{n-2}\left[\frac{2(m-l-1)(4a+2m-2l-1)(4b+2m-2l-1)}{(a+b+m-l-1)^2(2a+2b+2m-2l-3)}\right.$$
$$\left.\times\frac{(2a+2b+m-l-1)(a+b+m-l)}{(4a+4b+4m-4l-3)(4a+4b+4m-4l-1)}\right],\quad n\geq 3, \tag{6.27}$$

$$b_{m2n+1}=b_{m3}\prod_{l=1}^{n-1}\left[\frac{2(m-l)(4a+2m-2l-1)(4b+2m-2l-1)}{(a+b+m-l-1)(2a+2b+2m-2l-1)^2}\right.$$
$$\left.\times\frac{(2a+2b+m-l)(2a+2b+2m-2l+1)}{(4a+4b+4m-4l-1)(4a+4b+4m-4l+1)}\right],\quad n\geq 2. \tag{6.28}$$

It is clear that we can evaluate the $k$-level correlation functions for arbitrary $N$ and $k\leq N$ using the above results.

(2) The Legendre ensemble

Now we consider a limiting case, $a=b=0$. The monic skew Legendre polynomials of the first kind are known to be given by

$$Q_{2m}(x)=C_{2m}(x)+\frac{\sqrt{\pi}\,\Gamma(2m+1)}{2^{2m}\Gamma\left(2m+\frac{1}{2}\right)},$$

$$Q_{2m+1}(x)=C_{2m+1}(x)-\frac{2m(2m+1)}{(4m-1)(4m+1)}C_{2m-1}(x), \tag{6.29}$$

where

$$C_n(x)=2^n n!\frac{\Gamma(n+1)}{\Gamma(2n+1)}P_n^{(0,0)}(x). \tag{6.30}$$

Similarly to the case $\beta=1$, we can evaluate the global and local correlation functions. The results are as follows.[8,9]

(a) global $k$-level correlation functions:

$$G_4^{(k)}(x_1,\cdots,x_k)=\mathrm{T}\det[\lim_{N\to\infty}f(x_i,x_j)]_k, \tag{6.31}$$

where

$$f(x,y)=\begin{bmatrix}S(x,y) & I(x,y)\\ D(x,y) & S(y,x)\end{bmatrix},$$

$$S(x,y)=S_4(x,y),$$

$$I(x,y)=-\int_x^y S(x,y')\,dy',\quad D(x,y)=\frac{\partial}{\partial x}S(x,y), \tag{6.32}$$

with

$$\lim_{N\to\infty} S(\cos\theta, \cos\theta)=\frac{N}{\pi\sin\theta},$$

$$\lim_{N\to\infty} S(\cos\theta, \cos\varphi)=\frac{1}{2}\sqrt{\frac{N}{\pi}}\frac{1}{(\sin\varphi)^{3/2}}\sin\left\{\left(2N-\frac{1}{2}\right)\varphi-\frac{\pi}{4}\right\}, \quad 0<\theta, \varphi<\pi, \theta\neq\varphi. \tag{6.33}$$

The global 1-level correlation function (the level density) is

$$G_4^{(1)}(x)=\frac{N}{\pi\sqrt{1-x^2}}. \tag{6.34}$$

We find that this is identical to the level density formula (4.18) for $\beta=2$ and (5.34) for $\beta=1$.
(b) local $k$-level correlation functions:

$$L_4^{(k)}(w; \xi_1, \cdots, \xi_k)=\mathrm{T}\det\left[\lim_{N\to\infty}\frac{\pi\sqrt{1-w^2}}{N}f\left(w+\frac{\pi\sqrt{1-w^2}}{N}\xi_i, w+\frac{\pi\sqrt{1-w^2}}{N}\xi_j\right)\right]_k, \tag{6.35}$$

where

$$\lim_{N\to\infty}\frac{\pi\sqrt{1-w^2}}{N}S\left(w+\frac{\pi\sqrt{1-w^2}}{N}\xi, w+\frac{\pi\sqrt{1-w^2}}{N}\xi\right)=1,$$

$$\lim_{N\to\infty}\frac{\pi\sqrt{1-w^2}}{N}S\left(w+\frac{\pi\sqrt{1-w^2}}{N}\xi, w+\frac{\pi\sqrt{1-w^2}}{N}\eta\right)=\frac{\sin[2\pi(\xi-\eta)]}{2\pi(\xi-\eta)}, \quad -1<w<1.$$

(3) The Gaussian ensemble

We shall deal with the Gaussian ensemble at $\beta=4$. Firstly, we derive the monic skew Hermite polynomials of the first kind and their normalization constants $q_n$'s. From eqs. (2.15) and (6.11), we find that

$$q_n=2^{-4n-3/2}\sqrt{\pi}\,\Gamma(2n+2). \tag{6.36}$$

We expand the monic skew Hermite polynomials $Q_n(x)$ by the monic Hermite polynomials $C_n(x)$ as follows:

$$Q_{2m}(x)=a_{m0}C_{2m}(x)+a_{m2}C_{2m-2}(x)+a_{m4}C_{2m-4}(x)+\cdots,$$
$$Q_{2m+1}(x)=b_{m0}C_{2m+1}(x)+b_{m2}C_{2m-1}(x)+b_{m4}C_{2m-3}(x)+\cdots, \tag{6.37}$$

where

$$a_{m0}=b_{m0}=1,$$

and

$$C_n(x)=\frac{1}{2^{(3/2)n}}H_n(\sqrt{2}x). \tag{6.38}$$

We evaluate the coefficients $a_{mn}$'s and $b_{mn}$'s. As for the monic Hermite polynomials, the following property is known.

$$C_n'(x)=nC_{n-1}(x). \tag{6.39}$$

In addition, the monic Hermite polynomials have the following recurrence relations:

$$xC_n(x)=C_{n+1}(x)+\frac{n}{4}C_{n-1}(x). \tag{6.40}$$

Substituting $C_{2m-n}(x)$, $n\geq1$, and $Q_{2m}(x)$ for $f(x)$ and $g(x)$ in eq. (6.12) and using eqs. (6.39) and (6.40), we get

$$(2m-n+1)a_{mn-1}-4a_{mn+1}=0, \quad n\geq1. \tag{6.41}$$

Substituting $C_{2m-n+1}(x)$, $n\geq3$ and $Q_{2m+1}(x)$ for $f(x)$ and $g(x)$ in eq. (6.12) and using eqs. (6.39) and (6.40), we get

$$(2m-n+2)(2m-n+1)b_{mn-1}-4(2m-n)b_{mn+1}=0, \quad n\geq3. \tag{6.42}$$

Substituting $C_{2m}(x)$ and $Q_{2m+1}(x)$ for $f(x)$ and $g(x)$ in eq. (6.12), we get

$$b_{m2}=0. \tag{6.43}$$

Consequently, the monic skew Hermite polynomials of the first kind are given by

$$Q_{2m}(x)=C_{2m}(x)+\sum_{n=1}^{m}\left\{\frac{1}{2^n}\prod_{l=1}^{n}(m-l+1)\right\}C_{2m-2n}(x),$$
$$Q_{2m+1}(x)=C_{2m+1}(x). \tag{6.44}$$

Similarly to the case $\beta=1$, we can evaluate the global and local correlation functions. The results are as follows.

(a) global $k$-level correlation functions:

$$G_4^{(k)}(x_1,\cdots,x_k)=\mathrm{T}\det[\lim_{N\to\infty}f(x_i,x_j)]_k, \tag{6.45}$$

where

$$f(x,y)=\begin{bmatrix}S(x,y) & I(x,y)\\ D(x,y) & S(y,x)\end{bmatrix},$$
$$S(x,y)=S_4(x,y)+2yI_4(x,y),$$
$$I(x,y)=-\int_x^y S(x,y')\,dy',$$
$$D(x,y)=\frac{\partial}{\partial x}S(x,y), \tag{6.46}$$

with

$$\lim_{N\to\infty}S(x,x)=\frac{\sqrt{2N}}{\pi},\quad -R<x<R. \tag{6.47}$$

In this case we are again restricted to the vicinity of the origin and the global 1-level correlation function (the level density) is evaluated by the same physical argument as before. We get

$$G_4^{(1)}(x)=\frac{1}{\pi}\sqrt{2N-x^2},\quad x^2\le 2N$$
$$=0,\quad x^2\ge 2N. \tag{6.48}$$

This is identical to the particle density formula (4.26) for $\beta=2$ and (5.45) for $\beta=1$.

(b) local $k$-level correlation functions:

$$L_4^{(k)}(w;\xi_1,\cdots,\xi_k)=\mathrm{T}\det\left[\lim_{N\to\infty}\frac{\pi}{\sqrt{2N}}f\left(w+\frac{\pi}{\sqrt{2N}}\xi_i,\ w+\frac{\pi}{\sqrt{2N}}\xi_j\right)\right]_k, \tag{6.49}$$

where

$$\lim_{N\to\infty}\frac{\pi}{\sqrt{2N}}S\left(w+\frac{\pi}{\sqrt{2N}}\xi,\ w+\frac{\pi}{\sqrt{2N}}\xi\right)=1,$$
$$\lim_{N\to\infty}\frac{\pi}{\sqrt{2N}}S\left(w+\frac{\pi}{\sqrt{2N}}\xi,\ w+\frac{\pi}{\sqrt{2N}}\eta\right)=\frac{\sin[2\pi(\xi-\eta)]}{2\pi(\xi-\eta)},\quad -R<w<R.$$

(4) The Laguerre ensemble

We restrict ourselves to the case $a>0$. From eqs. (2.16) and (6.11), we find that

$$q_n=2^{-8a-8n-3}\Gamma(2n+2)\Gamma(4a+2n+1). \tag{6.50}$$

The monic skew Laguerre polynomials $Q_n(x)$ of the first kind can be expanded by the monic Laguerre polynomials $C_n(x)$ as follows:

$$Q_{2m}(x)=a_{m0}C_{2m}(x)+a_{m1}C_{2m-1}(x)+a_{m2}C_{2m-2}(x)+\cdots,$$
$$Q_{2m+1}=b_{m0}C_{2m+1}(x)+b_{m1}C_{2m}(x)+b_{m2}C_{2m-1}(x)+\cdots, \tag{6.51}$$

where

$$a_{m0}=b_{m0}=1,$$

and

$$C_n(x)=\frac{n!}{(-4)^n}L_n^{(4a)}(4x). \tag{6.52}$$

We evaluate the coefficients $a_{mn}$'s and $b_{mn}$'s. As for the monic Laguerre polynomials, the following property is known.

$$xC_n'(x)=nC_n(x)+\frac{n(n+4a)}{4}C_{n-1}(x). \tag{6.53}$$

In addition, the monic Laguerre polynomials have the following recurrence relations:

$$xC_n(x)=C_{n+1}(x)+\frac{4a+2n+1}{4}C_n(x)+\frac{n(4a+n)}{16}C_{n-1}(x). \tag{6.54}$$

Substituting $xC_{2m-n-1}(x)$, $n\geq 0$, and $Q_{2m}(x)$ for $f(x)$ and $g(x)$ in eq. (6.12) and using eqs. (6.53) and (6.54), we get a three-term recurrence relation for $a_{mn}$'s:

$$(2m-n)(4a+2m-n)a_{mn}-4a_{mn+1}-16a_{mn+2}=0. \quad n\geq 0. \tag{6.55}$$

Substituting $xC_{2m-n}(x)$, $n\geq 2$, and $Q_{2m+1}(x)$ for $f(x)$ and $g(x)$ in eq. (6.12) and using eqs. (6.53) and (6.54), we get a three-term recurrence relation for $b_{mn}$'s:

$$(2m-n+1)(4a+2m-n+1)b_{mn}-4b_{mn+1}-16b_{mn+2}=0. \quad n\geq 2. \tag{6.56}$$

By similar substitutions, we get

$$a_{m1}=\frac{m}{2}, \tag{6.57}$$

$$b_{m1}=c_m,\ b_{m2}=\frac{m}{8}(4c_m-2m-1),\ b_{m3}=\frac{m}{32}(4a+2m-1)(4c_m-2m-1). \tag{6.58}$$

Consequently, $a_{mn}$'s are given by

$$a_{m2n}=\prod_{l=1}^{n}\left[\frac{(m-l+1)(4a+2m-2l+1)}{8}\right], \quad n\geq 1, \tag{6.59}$$

$$a_{m2n+1}=a_{m1}\prod_{l=1}^{n}\left[\frac{(m-l)(4a+2m-2l+1)}{8}\right], \quad n\geq 1, \tag{6.60}$$

and $b_{mn}$'s are given by

$$b_{m2n}=b_{m2}\prod_{l=1}^{n-1}\left[\frac{(m-l)(4a+2m-2l+1)}{8}\right], \quad n\geq 2 \tag{6.61}$$

$$b_{m2n+1}=b_{m3}\prod_{l=1}^{n-1}\left[\frac{(m-l)(4a+2m-2l-1)}{8}\right], \quad n\geq 2. \tag{6.62}$$

It is clear that we can evaluate the $k$-level correlation functions for arbitrary $N$ and $k\leq N$ using the above results.

## §7. Discussion

(1) As far as we know, the local correlation functions of random matrix ensembles related to classical orthogonal polynomials depend only on the symmetry of the random matrices under consideration. They are invariant under the change of the ensembles, such as the values of the parameters $a$, $b$ or the position of measurement $w$. This means the nearest neighbour spacing distributions are also invariant under those changes since they are derived from the local correlation functions.

The local correlations are generally characterized by a function

$$f(x)=\frac{\sin(\pi x)}{\pi x}. \tag{7.1}$$

This function comes from the Christoffel-Darboux formula (4.11).

At $\beta=2$, the correlation functions $I_k$ are given by

$$I_k=\det\,[f(x_i, x_j)]_k. \tag{7.2}$$

The function $f(x_i, x_j)$ always takes the form of the Christoffel-Darboux formula regardless the orthogonal polynomial which corresponds to the ensemble. Therefore it is reasonable that the local correlation functions are generally expressed in terms of (7.1).

At $\beta=1$ or 4, the correlation functions $I_k$ are given by

$$I_k=\mathrm{T}\det\,[F(x_i, x_j)]_k, \tag{7.3}$$

and the quaternion function $F(x_i, x_j)$ is represented by a $2\times2$ matrix

$$F(x_i, x_j)=\begin{pmatrix} S(x_i, x_j) & -\int S(x_i, x_j)\,\mathrm{d}x_j \\ \dfrac{\partial}{\partial x_i}S(x_i, x_j) & S(x_j, x_i)\end{pmatrix}. \tag{7.4}$$

As for Legendre ensembles and Gaussian ensembles, we find that the dominant term of the function $S(x_i, x_j)$ takes the form of the Christoffel-Darboux formula. Hence the local correlation functions of both ensembles are characterized by the function (7.1). Is this property possessed by random matrix ensembles related to general orthogonal polynomials? Physically it is reasonable that the local correlation functions of one-dimensional statistical systems are independent of the external one-body potential. Mathematically, however, it is unclear why the dominant term of $S(x_i, x_j)$ takes the form of the Christoffel-Darboux formula.

As for the random matrix ensembles related to classical orthogonal polynomials, we can express the function $S(x_i, x_j)$ by using the corresponding orthogonal polynomials. Necessary formulae have been given in this paper. The next step is to investigate whether the dominant term in the thermodynamic limit is given by the Christoffel-Darboux formula or not. This is a crucial problem with regard to the universal behaviors of level fluctuations of random matrix ensembles.

(2) The 1-level correlation functions of Gaussian ensembles and Legendre ensembles are invariant in all the cases $\beta=1$, 2 and 4 in the thermodynamic limit. In the limit $\beta\to0$, the 1-level correlation functions of Gaussian ensembles and Legendre ensembles will become the Gaussian and the uniform distributions, respectively, because these ensembles are expected to be ideal gas systems. Therefore there is a kind of transition in both systems between $\beta=1$ and $\beta=0$. In the limit $\beta\to\infty$, the particles will rest on the zeros of corresponding polynomials.[15,16] It is an interesting problem to investigate the variation of distributions between $\beta=4$ and $\beta=\infty$.

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