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Correlations between Eigenvalues of a Random Matrix

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Abstract. Exact analytical expressions are found for the joint probability distribution functions of n eigenvalues belonging to a random Hermitian matrix of order N, where n is any integer and $N \to \infty$. The distribution functions, like those obtained earlier for $n = 2$, involve only trigonometrical functions of the eigenvalue differences.

I. Statement of Results

A finite stretch of eigenvalues $E_1, E_2, \ldots, E_r$ of a random Hermitian matrix H of order $N \gg r$ has a well-defined statistical behavior in the limit as $N \to \infty$. A convenient way to discuss this behavior is to relate the eigenvalues E_j to the angles θ_j belonging to a certain *Circular Ensemble* [1, 2]. If D is the mean level-spacing of the eigenvalue series, we write

$$\theta_j = \frac{2\pi}{ND} E_j, \quad j = 1, \ldots, r, \tag{1.1}$$

and take for the complete series of angles $(\theta_1, \ldots, \theta_N)$ the probability distribution

$$Q_{N\beta}(\theta_1, \ldots, \theta_N) = C_{N\beta} \prod_{j<k} |e^{i\theta_j} - e^{i\theta_k}|^\beta, \tag{1.2}$$

where $\beta = 1, 2$ or 4. The case $\beta = 1$ applies to the usual physical situation in which H is real and symmetric, in particular when H is invariant under time-reflection and under space-rotations. The case $\beta = 2$ would apply when H is complex Hermitian, i.e. when there is no time-reflection invariance. The case $\beta = 4$ would apply when H is invariant under time-reflection, without any rotation-invariance, for a system with half-integer spin. Until now no interesting physical examples have been found of the cases $\beta = 2$ and 4. The case $\beta = 1$ has been extensively studied in connection with the statistics of neutron capture levels in heavy nuclei [3–6].

The distribution-functions $Q_{N\beta}$ are normalized so that

$$Q_{N\beta}(\theta_1, \ldots, \theta_N)\, d\theta_1 \ldots d\theta_N \tag{1.3}$$

is the probability of finding one angle, regardless of labelling, within each of the intervals $[\theta_j, \theta_j + d\theta_j]$. We have then

$$\int \cdots \int_0^{2\pi} Q_{N\beta}(\theta_1, \ldots, \theta_N)\, d\theta_1 \ldots d\theta_N = N!\,, \tag{1.4}$$

with the normalization constants [1]

$$C_{N1} = 2^{-N} \pi^{-\frac{1}{2}(N+1)} \Gamma(\tfrac{1}{2} + \tfrac{1}{2}N)\,, \tag{1.5}$$

$$C_{N2} = (2\pi)^{-N}\,, \tag{1.6}$$

$$C_{N4} = \pi^{-N}(N!/(2N)!)\,. \tag{1.7}$$

The n-angle correlation function $R_{Nn\beta}$ is defined by

$$\begin{aligned} R_{Nn\beta}(\theta_1, \ldots, \theta_n) &= (1/(N-n)!) \\ &\quad \times \int \cdots \int_0^{2\pi} d\theta_{n+1} \ldots d\theta_N Q_{N\beta}(\theta_1, \ldots, \theta_N)\,. \end{aligned} \tag{1.8}$$

This gives the probability density for finding n angles at the positions $(\theta_1, \ldots, \theta_n)$, regardless of the positions of the remaining angles. In particular, for the circular ensembles

$$R_{N0\beta} = 1, \quad R_{N1\beta}(\theta_1) = (N/2\pi)\,. \tag{1.9}$$

The n-level correlation-function $P_{n\beta}$ of the eigenvalue series E_j is defined by

$$P_{n\beta}(E_1, \ldots, E_n) = \lim_{N\to\infty} \left(\frac{2\pi}{ND}\right)^n R_{Nn\beta}(\theta_1, \ldots, \theta_n)\,, \tag{1.10}$$

with the θ_j given by Eq. (1.1). The statistical properties of the eigenvalues are completely characterized by the functions $P_{n\beta}$.

We have previously calculated the two-level correlations $P_{2\beta}$, and the n-level correlation $P_{n\beta}$ for $\beta = 2$. The results were as follows [2, 7]. Write

$$s(r) = (\sin(\pi r)/(\pi r))\,, \tag{1.11}$$

$$Ds(r) = (ds(r)/dr)\,, \tag{1.12}$$

$$Is(r) = \int_0^r s(r')\,dr'\,, \tag{1.13}$$

$$Js(r) = Is(r) - \varepsilon(r)\,, \tag{1.14}$$

where $\varepsilon(r)$ is the step-function

$$\begin{aligned} \varepsilon(r) &= \tfrac{1}{2}, \quad (r>0), \\ &= 0, \quad (r=0), \\ &= -\tfrac{1}{2}, \quad (r<0). \end{aligned} \tag{1.15}$$

Then

$$P_{21}(E_1, E_2) = D^{-2}[1-(s(r))^2 + Js(r).\, Ds(r)], \tag{1.16}$$

$$P_{22}(E_1, E_2) = D^{-2}[1-(s(r))^2], \tag{1.17}$$

$$P_{24}(E_1, E_2) = D^{-2}[1-(s(2r))^2 + Is(2r).\, Ds(2r)], \tag{1.18}$$

with

$$r = ((E_1 - E_2)/D). \tag{1.19}$$

Also

$$P_{n2}(E_1, \ldots, E_n) = D^{-n}\,\mathrm{Det}[s(r_{ij})]_{i,j=1,\ldots,n}, \tag{1.20}$$

with

$$r_{ij} = ((E_i - E_j)/D). \tag{1.21}$$

In the present paper we complete the determination of eigenvalue correlations by finding explicit formulae for all the $P_{n\beta}$ with $\beta = 1, 4$. The formulae turn out to be surprisingly compact and are well adapted for practical use. The derivation of these results also gives a better insight into the peculiar structure of the two-level correlation-functions (1.16) and (1.18).

To state our conclusions it is convenient to use the word *quaternion* as a synonym for a (2×2) matrix with real or complex coefficients,

$$q = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{1.22}$$

The quaternion units are

$$X = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \tag{1.23}$$

and the quaternion adjoint to q is

$$\bar{q} = (\mathrm{Tr}\, q)I - q = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{1.24}$$

We shall be concerned with an $(N\times N)$ matrix M whose elements M_{ij} are themselves (2×2) matrices. To avoid confusion of language we refer to the M_{ij} as quaternions rather than matrices. The matrix M is defined to be *self-dual* if

$$M_{ji} = \bar{M}_{ij}. \tag{1.25}$$

Let M be a self-dual matrix of quaternions. Then we can define the *quaternion-determinant*

$$\mathrm{Q\,Det}\,M=\sum_P(-1)^{N-l}\prod_1^l(M_{ab}M_{bc}\dots M_{sa})\,. \tag{1.26}$$

Here P is any permutation of the integers $(1,2,\dots,N)$, consisting of l cycles of the form

$$(a\to b\to c\to\cdots\to s\to a)\,, \tag{1.27}$$

and

$$(-1)^{N-l} \tag{1.28}$$

is the parity of P. In words, $\mathrm{Q\,Det}\,M$ is obtained from the ordinary expression for the determinant of M by arranging the factors in each monomial in an order determined by the cyclic operation of the corresponding permutation P. In particular, if the elements of M are scalars, $\mathrm{Q\,Det}\,M$ reduces to the ordinary determinant $\mathrm{Det}\,M$.

The definition (1.26) is not yet complete, because the value of the product on the right-hand side may depend on the order in which the l cyclic factors are written. To make the definition unique, we require that the same ordering of the l cyclic factors be used for the permutation P and for the other permutations obtained from P by reversing the direction of some or all of the cycles (1.27). Since M is self-dual,

$$(M_{as}\dots M_{cb}M_{ba})=\overline{(M_{ab}M_{bc}\dots M_{sa})}\,. \tag{1.29}$$

Thus in the sum (1.26) we may replace each factor $(M_{ab}M_{bc}\dots M_{sa})$ by

$$\tfrac{1}{2}(M_{ab}M_{bc}\dots M_{sa}+M_{as}\dots M_{cb}M_{ba})=\tfrac{1}{2}\mathrm{Tr}\,(M_{ab}M_{bc}\dots M_{sa})\,, \tag{1.30}$$

by virtue of Eq. (1.24) and (1.29). Therefore the value of Eq. (1.26) after summing over P is independent of the order of the l cyclic factors. Also $\mathrm{Q\,Det}\,M$ is a scalar. Strictly speaking, we should define $\mathrm{Q\,Det}\,M$ for non-self-dual M by inserting the operation $(\tfrac{1}{2}\mathrm{Tr})$ before each cyclic product in Eq. (1.26). However, we shall be concerned only with self-dual M, and for these the definition (1.26) as it stands is preferable.

For $\beta=1,4$ we define the function $\sigma_\beta(r)$ as a quaternion with the $[2\times 2]$ matrix representation

$$\sigma_1(r)=\begin{bmatrix}s(r) & Ds(r)\\ Js(r) & s(r)\end{bmatrix}, \tag{1.31}$$

$$\sigma_4(r)=\begin{bmatrix}s(2r) & Ds(2r)\\ Is(2r) & s(2r)\end{bmatrix}, \tag{1.32}$$

the matrix elements being given by Eq. (1.11)–(1.15). For $\beta = 2$ we take $\sigma_\beta(r)$ to be the scalar

$$\sigma_2(r) = s(r). \tag{1.33}$$

Our main result is then

Theorem 1. *The n-level correlation-function for eigenvalues defined by the ensemble* (1.2) *in the limit* $N \to \infty$ *is*

$$P_{n\beta}(E_1, \ldots, E_n) = D^{-n}\, \mathrm{QDet}[\sigma_\beta(r_{ij})]_{i,j=1,\ldots,n}, \tag{1.34}$$

with σ_β *defined by Eq.* (1.31)–(1.33) *and* r_{ij} *by Eq.* (1.21).

Remark 1. The quaternion matrix $[\sigma_\beta(r_{ij})]$ is self-dual, since the function $s(r)$ is even in r while $Ds(r)$, $Js(r)$ and $Is(r)$ are odd. Therefore $P_{n\beta}$ is a scalar.

Remark 2. Theorem 1 includes as special cases Eq. (1.16)–(1.20).

Remark 3. Theorem 1 can be further simplified by restating it in terms of the n-level Cluster-functions [7], which are defined by

$$P_{n\beta}(E_1, \ldots, E_n) = \sum_G (-1)^{n-l} \prod_{t=1}^{l} (Y_{h(t),\beta}(E_j; j \in G_t)). \tag{1.35}$$

Here G denotes any division of the indices $(1, \ldots, n)$ into unordered subsets $(G_1, \ldots, G_l)$, $h(t)$ is the number of indices in G_t, and $Y_{n\beta}$ is the n-level cluster-function. The determinant (1.26) is precisely of the form (1.35), and therefore

$$Y_{n\beta}(E_1, \ldots, E_n) = \sum_P [\sigma_\beta(r_{12})\, \sigma_\beta(r_{23}) \ldots \sigma_\beta(r_{n1})], \tag{1.36}$$

where $\sum_P$ denotes a sum over the $(n-1)!$ distinct cyclic permutations of the indices $(1, 2, \ldots, n)$. Like $P_{n\beta}$, $Y_{n\beta}$ is a scalar, and its scalar character can be made explicit for $\beta = 1, 4$ by inserting the operation $(\frac{1}{2}\mathrm{Tr})$ before the cyclic product in Eq. (1.36). The cluster-function $Y_{n\beta}$ describes those correlations in a cluster of n levels which are additional to the effects of correlations in clusters of $m < n$ levels.

Remark 4. In practical applications of the theory [5], it is most convenient to work with the Fourier transforms of the cluster-functions. We write

$$\begin{aligned} y_{n\beta}(k_1, \ldots, k_n)\, \delta(k_1 + \cdots + k_n) = \int \cdots \int_{-\infty}^{\infty} & dE_1 \ldots dE_n Y_{n\beta}(E_1, \ldots, E_n) \\ & \cdot \exp\left[(2\pi i/D) \sum_{j=1}^{n} E_j k_j\right]. \end{aligned} \tag{1.37}$$

Let then

$$f(k) = 1 \quad (|k| < \tfrac{1}{2}), \tag{1.38}$$

$$f(k) = 0 \quad (|k| > \tfrac{1}{2}), \tag{1.39}$$

$$g(k) = 1 - f(k), \tag{1.40}$$

$$\tilde{\sigma}_1(k) = \begin{bmatrix} f(k) & kf(k) \\ -k^{-1}g(k) & f(k) \end{bmatrix}, \tag{1.41}$$

$$\tilde{\sigma}_2(k) = f(k), \tag{1.42}$$

$$\tilde{\sigma}_4(k) = \tfrac{1}{2} f(\tfrac{1}{2}k) \begin{bmatrix} 1 & k \\ k^{-1} & 1 \end{bmatrix}. \tag{1.43}$$

Some factors $(i, -i)$ which do not affect the value of $y_{n\beta}$ have here been dropped.

Eq. (1.36) gives

$$\begin{aligned} y_{n\beta}(k_1, \ldots, k_n) = \int_{-\infty}^{\infty} dp \sum_P \\ \times [\tilde{\sigma}_\beta(p)\, \tilde{\sigma}_\beta(p + k_1) \ldots \tilde{\sigma}_\beta(p + k_1 + \cdots + k_{n-1})] . \end{aligned} \tag{1.44}$$

The single integration in Eq. (1.44) gives at worst a rational-logarithmic function of the variables $(k_1, \ldots, k_n)$.

The following sections of this paper will be occupied with the proof of Theorem 1.

II. Quaternion-Determinants

To every $(N \times N)$ quaternion-matrix M corresponds an ordinary $(2N \times 2N)$ matrix $A(M)$ which is obtained by regarding each element M_{ij} of M as a $[2 \times 2]$ block of matrix elements in $A(M)$. The operation $A(\ )$ commutes with the matrix operations of addition and multiplication. For M to be self-dual, it is necessary and sufficient that

$$[A(M)]^T = Y A(M) Y^{-1}, \tag{2.1}$$

where T denotes transposition and Y is the quaternion unit given by Eq. (1.23). The basic property of quaternion-determinants is expressed in

Theorem 2. *For any self-dual quaternion matrix M,*

$$[\mathrm{Q\,Det}\, M]^2 = \mathrm{Det}[A(M)] . \tag{2.2}$$

Remark 1. When M is self-dual, Eq. (2.1) shows that the matrix

$$B(M) = -YA(M) \tag{2.3}$$

is antisymmetric. We have then

$$\mathrm{Q\,Det}\,M = \mathrm{Pf}[B(M)], \tag{2.4}$$

where Pf denotes the Pfaffian. Theorem 2 is merely a restatement of the well-known property of Pfaffians [8]

$$[\mathrm{Pf}\,B]^2 = \mathrm{Det}\,B. \tag{2.5}$$

An elegant proof of Eq. (2.4) has been found by Balian and Brézin [9]. Here, instead of using Eq. (2.4)–(2.5), we prove Theorem 2 directly.

Remark 2. Theorem 2 is essentially a restatement in more convenient notation of the theorem of Mehta ([2], Appendix A.7, p. 194) on the expansion of a Pfaffian.

Proof of Theorem 2. The Quaternion-matrix L adjoint to M is defined by

$$L_{ij} = \sum_{P'} (-1)^{N-l} \left\{ \prod_{1}^{l-1} (M_{ab} M_{bc} \dots M_{sa}) \right\} \quad (M_{ie} M_{ef} \dots M_{tj}), \tag{2.6}$$

where P' is restricted to permutations of $(1, 2, \dots, N)$ such that

$$P'(j) = i, \tag{2.7}$$

and the cycle of P' containing i and j is

$$(i \to e \to f \to \cdots \to t \to j \to i). \tag{2.8}$$

The value of L_{ij} is independent of the order of the l cyclic factors in Eq. (2.6), when the sum over P' is carried out according to the same rule as was used for Eq. (1.26). Comparison of Eq. (2.6) with (1.26) gives for any self-dual M

$$ML = LM = (\mathrm{Q\,Det}\,M) I_N, \tag{2.9}$$

where I_N is the $(N \times N)$ unit quaternion matrix. In $(2N \times 2N)$ matrix notation, Eq. (2.9) becomes

$$A(L)\,A(M) = (\mathrm{Q\,Det}\,M) I_{2N}. \tag{2.10}$$

Suppose now $\mathrm{Det}\,A(M) = 0$. Then there exists a non-zero $2N$-component vector Λ with

$$A(M)\Lambda = 0, \tag{2.11}$$

and Eq. (2.10) implies $\mathrm{Q\,Det}\,M = 0$. Thus $\mathrm{Q\,Det}\,M = 0$ whenever $\mathrm{Det}\,A(M) = 0$. But $\mathrm{Q\,Det}\,M$ is a multilinear polynomial in the matrix elements of $A(M)$ with leading term

$$M_{11} M_{22} \ldots M_{NN}, \tag{2.12}$$

whereas $\mathrm{Det}\,A(M)$ is a multiquadratic polynomial with leading term

$$M_{11}^2 M_{22}^2 \ldots M_{NN}^2. \tag{2.13}$$

Since $\mathrm{Q\,Det}\,M$ is symmetric under permutations of the indices $(1, \ldots, N)$, it must either be irreducible or else be a product of N linear factors each containing one of the M_{jj}. In either case, only the same irreducible factors can occur in $\mathrm{Det}\,A(M)$. By Eq. (2.13) each factor must occur squared, and Eq. (2.2) is proved.

III. Eigenvalue Distributions on a Circle

We prove Theorem 1 by finding explicit expressions for the correlation-functions $R_{Nn\beta}$ defined by Eq. (1.8). Let N be any positive integer. We write

$$s_N(\theta) = \frac{\sin(\frac{1}{2}N\theta)}{2\pi \sin(\frac{1}{2}\theta)} = \frac{1}{2\pi} \sum_p e^{ip\theta}, \tag{3.1}$$

where p takes the values

$$p = \tfrac{1}{2}(1-N), \tfrac{1}{2}(3-N), \ldots, \tfrac{1}{2}(N-3), \tfrac{1}{2}(N-1). \tag{3.2}$$

The values of p are integral if N is odd, half-integral if N is even. The function $s_N(\theta)$ is even in θ, and

$$s_N(\theta + 2\pi) = (-1)^{N-1} s_N(\theta). \tag{3.3}$$

We write

$$D s_N(\theta) = (d/d\theta)\, s_N(\theta) = \frac{1}{2\pi} \sum_p i p e^{ip\theta}, \tag{3.4}$$

and

$$I s_N(\theta) = \int_0^\theta s_N(\theta') d\theta', \tag{3.5}$$

so that

$$I s_N(\theta) = \frac{1}{2\pi i} \sum_p p^{-1} e^{ip\theta}, \qquad N \text{ even} \tag{3.6}$$

$$I s_N(\theta) = \frac{1}{2\pi i} \sum_{p \neq 0} p^{-1} e^{ip\theta} + \frac{1}{2\pi}\theta, \qquad N \text{ odd}. \tag{3.7}$$

For all N we write

$$J s_N(\theta) = -\frac{1}{2\pi i}\sum_q q^{-1} e^{iq\theta}, \tag{3.8}$$

where q takes the values

$$q = \pm\tfrac{1}{2}(N+1),\ \pm\tfrac{1}{2}(N+3), \ldots \tag{3.9}$$

Then

$$I s_N(\theta) - J s_N(\theta) = \varepsilon_N(\theta) \tag{3.10}$$

is a step-function whose character depends only on the parity of N. In fact, for any integer m with

$$2\pi m < \theta < 2\pi(m+1), \tag{3.11}$$

we have

$$\varepsilon_N(\theta) = \tfrac{1}{2}(-1)^m, \qquad N \text{ even}, \tag{3.12}$$

$$\varepsilon_N(\theta) = m + \tfrac{1}{2}, \qquad N \text{ odd}. \tag{3.13}$$

At the points of discontinuity $\theta = 2\pi m$,

$$\varepsilon_N(\theta) = 0, \qquad (N \text{ even}), \tag{3.14}$$

$$\varepsilon_N(\theta) = m, \qquad (N \text{ odd}). \tag{3.15}$$

The lack of uniform convergence of the series defining $J s_N$ will not cause any difficulty. The functions $D s_N$, $I s_N$, $J s_N$ and ε_N are all odd in θ.

We define the quaternions $\sigma_{N\beta}(\theta)$ for $\beta = 1, 4$ by their matrix representations

$$\sigma_{N1}(\theta) = \begin{bmatrix} s_N(\theta) & D s_N(\theta) \\ J s_N(\theta) & s_N(\theta) \end{bmatrix}, \tag{3.16}$$

$$\sigma_{N4}(\theta) = \frac{1}{2}\begin{bmatrix} s_{2N}(\theta) & D s_{2N}(\theta) \\ I s_{2N}(\theta) & s_{2N}(\theta) \end{bmatrix}. \tag{3.17}$$

For $\beta = 2$, $\sigma_{N\beta}$ is the scalar

$$\sigma_{N2}(\theta) = s_N(\theta). \tag{3.18}$$

We shall study the quaternion-determinants

$$U_{Nn\beta}(\theta_1, \ldots, \theta_n) = \mathrm{Q\,Det}\,[\sigma_{N\beta}(\theta_i - \theta_j)]_{i,j=1,\ldots,n}, \tag{3.19}$$

which are functions of n angles $(\theta_1, \ldots, \theta_n)$.

In this and the following section we prove

Theorem 3. *For* $\beta = 1, 2, 4,$

$$U_{NN\beta}(\theta_1, \ldots, \theta_N) = C_{N\beta}|\Delta|^{\beta}, \tag{3.20}$$

with

$$\Delta = \prod_{j<k} (e^{i\theta_j} - e^{i\theta_k}), \tag{3.21}$$

and $C_{N\beta}$ *given by Eq.* (1.5)–(1.7).

Remark 1. Theorem 3 states that $U_{NN\beta}$ is the normalized joint probability distribution for the angles $(\theta_1, \ldots, \theta_N)$ in the circular ensemble discussed in Section I.

Remark 2. The case $\beta = 2$ is well-known and simple to prove.

Remark 3. The most difficult and interesting case of Theorem 3 is $\beta = 1$. In this case Theorem 3 shows that the use of a quaternion-determinant allows us to take the "positive square-root" of the symmetric determinant $\mathrm{Det}[s_N(\theta_i - \theta_j)]$. Previously the use of Pfaffians was restricted to taking square-roots of antisymmetric determinants.

Proof of Theorem 3. The case $\beta = 2$ being trivial, we suppose henceforth that $\beta = 1$ or 4. $U_{NN\beta}$ is then the quaternion-determinant of a self-dual matrix, and Eq. (2.2) gives

$$(U_{NN\beta})^2 = \mathrm{Det}\, A(\sigma_{N\beta}(\theta_i - \theta_j)), \tag{3.22}$$

where $A(\sigma_{N\beta})$ is the $[2N \times 2N]$ matrix specified by Eq. (3.16), (3.17).

Consider first the case $\beta = 1$, N even.

The $(2N \times 2N)$ matrix product

$$\begin{aligned} P &= \frac{1}{2\pi} \begin{bmatrix} e^{ip\theta_j} & 0 \\ (ip)^{-1} e^{ip\theta_j} & 0 \end{bmatrix} \begin{bmatrix} e^{-ip\theta_k} & ipe^{-ip\theta_k} \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} s_N(\theta_j - \theta_k) & Ds_N(\theta_j - \theta_k) \\ Is_N(\theta_j - \theta_k) & s_N(\theta_j - \theta_k) \end{bmatrix} \end{aligned} \tag{3.23}$$

has rank N, since the first factor has all even-numbered columns zero and the second factor has all even-numbered rows zero. Therefore the value of $\mathrm{Det}\, A(\sigma_{N1}(\theta_j - \theta_k))$ is not changed when we subtract the even-numbered rows of P from the corresponding rows of $A(\sigma_{N1})$. The subtraction gives

$$\mathrm{Det}\, A(\sigma_{N1}) = \mathrm{Det}[\varepsilon_N(\theta_j - \theta_k)].\, \mathrm{Det}[Ds_N(\theta_j - \theta_k)], \tag{3.24}$$

by virtue of Eq. (3.10). Now Eq. (3.4) and (3.21) imply

$$\begin{aligned} \mathrm{Det}[Ds_N(\theta_j - \theta_k)] &= (2\pi)^{-N} i^N \prod_p p\; |\mathrm{Det}[e^{ip\theta_j}]|^2 \\ &= 2^{-N} \pi^{-N-1} (\Gamma(\tfrac{1}{2} + \tfrac{1}{2}\mathrm{N}))^2\, |\Delta|^2 . \end{aligned} \tag{3.25}$$

The quantity

$$d_N = \mathrm{Det}[\varepsilon_N(\theta_j - \theta_k)] \tag{3.26}$$

is (i) piecewise constant with possible discontinuities only at places where $\theta_j - \theta_k = 2\pi m$ with integer m, (ii) periodic with period 2π in each variable θ_j, and (iii) a symmetric function of $(\theta_1, \ldots, \theta_N)$. It follows from these three properties that d_N must be a constant independent of $(\theta_1, \ldots, \theta_N)$, except at the points of discontinuity where $\Delta = 0$. Therefore we may take $d_N =$ constant in Eq. (3.24). To evaluate d_N we take

$$2\pi > \theta_1 > \theta_2 > \cdots > \theta_N > 0\,. \tag{3.27}$$

Then

$$d_N = 2^{-N} \begin{vmatrix} 0 & 1 & 1 \ldots 1 \\ -1 & 0 & 1 \ldots 1 \\ \vdots & \vdots & \vdots \quad \vdots \\ -1 & -1 & -1 \ldots 0 \end{vmatrix} = 2^{-N}\,. \tag{3.28}$$

Eq. (1.5), (3.24), (3.25), and (3.28) give for $\beta = 1$ and N even

$$\mathrm{Det}\, A[\sigma_{N\beta}(\theta_j - \theta_k)] = (C_{N\beta})^2\, |\Delta|^{2\beta}\,. \tag{3.29}$$

Next consider $\beta = 1$, N odd.

In this case zero appears as a value of p in Eq. (3.2), and P cannot be defined by Eq. (3.23). Let P_δ be the matrix product obtained from P by the replacements

$$(ip)^{-1} e^{ip\theta_j} \to \delta^{-1}, \qquad ip\, e^{-ip\theta_k} \to \delta\,, \tag{3.30}$$

for the elements with $p = 0$, where δ is any non-zero quantity. Then

$$P_\delta = \begin{bmatrix} s_N(\theta_j - \theta_k) & D s_N(\theta_j - \theta_k) + (\delta/2\pi) \\ I s_N(\theta_j - \theta_k) + (1/2\pi)(\delta^{-1} - (\theta_j - \theta_k)) & s_N(\theta_j - \theta_k) \end{bmatrix} \tag{3.31}$$

is still of rank N. Instead of $A(\sigma_{N1}(\theta_j - \theta_k))$ we consider the matrix

$$A_\delta = \begin{bmatrix} s_N(\theta_j - \theta_k) & D s_N(\theta_j - \theta_k) + (\delta/2\pi) \\ J s_N(\theta_j - \theta_k) & s_N(\theta_j - \theta_k) \end{bmatrix}. \tag{3.32}$$

The determinant of A_δ is unchanged by subtraction of the even-numbered rows of P_δ from those of A_δ. Therefore

$$\begin{aligned} \mathrm{Det}\, A_\delta = {} & \mathrm{Det}[\varepsilon_N(\theta_j - \theta_k) + ((\theta_k - \theta_j + \delta^{-1})/2\pi)] \\ & \times \mathrm{Det}[D s_N(\theta_j - \theta_k) + (\delta/2\pi)]\,. \end{aligned} \tag{3.33}$$

The second factor on the right of Eq. (3.33) is

$$(2\pi)^{-N} i^{N-1} \Big(\prod_{p \neq 0} p\Big) \delta |\mathrm{Det}(e^{ip\theta_j})|^2 = (2\pi)^{-N} (\Gamma(\tfrac{1}{2}+\tfrac{1}{2}N))^2 \, \delta |\Delta|^2 . \tag{3.34}$$

In the first factor we subtract the first column from each of the remaining columns, obtaining

$$(2\pi\delta)^{-1} \,\mathrm{Det}[1_N + O(\delta), \varepsilon_N(\theta_j - \theta_k) - \varepsilon_N(\theta_j - \theta_1) + ((\theta_k - \theta_1)/2\pi)] , \tag{3.35}$$

where 1_N means a single column of unit elements, and k labels the remaining columns from 2 to N. We can now pass to the limit $\delta \to 0$ in Eq. (3.32), (3.33). We obtain

$$\mathrm{Det}\, A(\sigma_{N1}) = (2\pi)^{-N-1} (\Gamma(\tfrac{1}{2}+\tfrac{1}{2}N))^2 \, d_N |\Delta|^2 , \tag{3.36}$$

where now

$$d_N = \mathrm{Det}[1_N, \varepsilon_N(\theta_j - \theta_k) - \varepsilon_N(\theta_j - \theta_1)] , \tag{3.37}$$

the terms $((\theta_k - \theta_1)/2\pi)$ in Eq. (3.35) contributing nothing to the determinant. By the same argument as was used for N even, d_N is a constant independent of $(\theta_1, \ldots, \theta_N)$ except at places where $\Delta = 0$. The value of d_N is found by taking the θ_j to satisfy Eq. (3.27) and is

$$d_N = 2^{1-N} . \tag{3.38}$$

Therefore Eq. (3.29) holds also for $\beta = 1$ and N odd.

Finally we have the case $\beta = 4$.

The matrix $A(\sigma_{N4})$ is a product

$$A(\sigma_{N4}) = \frac{1}{4\pi} \begin{bmatrix} e^{ip\theta_j} \\ (ip)^{-1} e^{ip\theta_j} \end{bmatrix} [e^{-ip\theta_k}, ipe^{-ip\theta_k}] , \tag{3.39}$$

where now the index p takes the $2N$ values

$$p = \tfrac{1}{2} - N, \tfrac{3}{2} - N, \ldots, N - \tfrac{1}{2} . \tag{3.40}$$

Therefore

$$\mathrm{Det}\, A(\sigma_{N4}) = (4\pi)^{-2N} (-i)^{2N} \Big(\prod_p p^{-1}\Big) |\Delta'|^2 = (C_{N4})^2 |\Delta'|^2 , \tag{3.41}$$

by virtue of Eq. (1.7), where

$$\Delta' = \mathrm{Det}[e^{ip\theta_j}, pe^{ip\theta_j}] \tag{3.42}$$

is the Confluent Alternant discussed by Mehta ([2], Appendix A.16, p. 208). According to Mehta

$$|\Delta'| = |\Delta|^4 , \tag{3.43}$$

and so Eq. (3.29) holds also for $\beta = 4$.

Eq. (3.22) and (3.29) imply

$$U_{NN\beta} = \eta_{N\beta} C_{N\beta} |\Delta|^{\beta}, \tag{3.44}$$

where $\eta_{N\beta} = \pm 1$. The sign of $\eta_{N\beta}$ might still depend on the θ_j. However, both sides of Eq. (3.44) are (i) symmetric functions of $(\theta_1, \ldots, \theta_N)$, (ii) periodic in each θ_j with period 2π, and (iii) continuous functions of θ_j except at places where $\Delta = 0$. Therefore $\eta_{N\beta}$ is $+1$ or -1 independent of $(\theta_1, \ldots, \theta_N)$. This completes the proof of Theorem 3, except for the determination of the sign of $\eta_{N\beta}$ which we postpone to the following section.

IV. Eigenvalue Correlations

Our final task is to prove

Theorem 4. *For* $1 \leq n \leq N$ *and* $\beta = 1, 2, 4$,

$$R_{Nn\beta} = U_{Nn\beta}, \tag{4.1}$$

where $R_{Nn\beta}$ *is the n-angle correlation function defined by Eq.* (1.8), *and* $U_{Nn\beta}$ *is the quaternion-determinant defined by Eq.* (3.19).

Remark 1. Theorem 1 follows immediately from Theorem 4 by taking the limit $N \to \infty$ and using Eq. (1.1), (1.10).

Remark 2. Theorem 3 is the special case $n = N$ of Theorem 4. We shall deduce Theorem 4 from Theorem 3, verifying incidentally that $\eta_{N\beta} = +1$ in Eq. (3.44).

Proof of Theorem 4. We consider the functions

$$V_{Nn\beta}(\theta_1, \ldots, \theta_n) = \sum_P [\sigma_{N\beta}(\theta_1 - \theta_2) \ldots \sigma_{N\beta}(\theta_n - \theta_1)], \tag{4.2}$$

with P summed over cyclic permutations of $(1, \ldots, n)$ as in Eq. (1.36). The $\sigma_{N\beta}$ are defined by Eq. (3.16)–(3.18), and $V_{Nn\beta}$ is therefore a scalar. The $U_{Nn\beta}$ and $V_{Nn\beta}$ are related by

$$U_{Nn\beta}(\theta_1, \ldots, \theta_n) = \sum_G (-1)^{n-l} \prod_{t=1}^{l} (V_{Nh(t)\beta}(\theta_j; j \in G_t)), \tag{4.3}$$

like the $P_{n\beta}$ and $Y_{n\beta}$ in Eq. (1.35). Theorem 4 states that the $U_{Nn\beta}$ are the correlation-functions for the distribution (1.2), and this is equivalent to the statement that the $V_{Nn\beta}$ are the cluster-functions for the same distribution.

For any two functions $f_1(\theta)$, $f_2(\theta)$, we define the composition

$$(f_1 * f_2)(\theta) = \int_0^{2\pi} d\varphi\, f_1(\varphi) f_2(\theta - \varphi). \tag{4.4}$$

Then the definitions (3.1)–(3.8) give

$$s_N * s_N = s_N \,, \tag{4.5}$$

$$D s_N * s_N = s_N * D s_N = D s_N \,, \tag{4.6}$$

$$J s_N * s_N = s_N * J s_N = 0 \,, \tag{4.7}$$

$$J s_N * D s_N = D s_N * J s_N = 0 \,, \tag{4.8}$$

and for N even only,

$$I s_N * s_N = s_N * I s_N = I s_N \,, \tag{4.9}$$

$$I s_N * D s_N = D s_N * I s_N = s_N \,. \tag{4.10}$$

The definition (4.4) applies equally to the composition-product of two quaternions. Thus Eq. (3.16) with (4.4)–(4.8) gives

$$\sigma_{N1} * \sigma_{N1} = \begin{bmatrix} s_N & 2 D s_N \\ 0 & s_N \end{bmatrix} = \sigma_{N1} + E \sigma_{N1} - \sigma_{N1} E \,, \tag{4.11}$$

where

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} . \tag{4.12}$$

On the other hand, Eq. (3.17) and (3.18) with Eq. (4.5)–(4.10) give simply

$$\sigma_{N\beta} * \sigma_{N\beta} = \sigma_{N\beta}, \qquad \beta = 2, 4 \,. \tag{4.13}$$

Let $V_{Nn\beta}(\theta_1, \ldots, \theta_n)$ given by Eq. (4.2) be integrated with respect to θ_n from 0 to 2π. After making use of Eq. (4.11) or (4.13), we obtain two kinds of terms, those involving E and those not involving E. The E-terms cancel each other exactly after summing over permutations. The non-E terms give precisely the terms which appear in $V_{N,n-1,\beta}(\theta_1, \ldots, \theta_{n-1})$, each repeated $(n-1)$ times, since every cyclic permutation of $(1, \ldots, n)$ can be obtained in $(n-1)$ ways by inserting n into a cyclic permutation of $(1, \ldots, n-1)$. Therefore for $n = 2, 3, \ldots, N$,

$$\int_0^{2\pi} V_{Nn\beta}(\theta_1, \ldots, \theta_n) d\theta_n = (n-1)\, V_{N,n-1,\beta}(\theta_1, \ldots, \theta_{n-1}) \,. \tag{4.14}$$

This is precisely the recurrence relation between cluster-functions (see Dyson [7]). On the other hand, for $n = 1$ we have trivially

$$\int_0^{2\pi} V_{N1\beta}(\theta_1) d\theta_1 = N \,. \tag{4.15}$$

When Eq. (4.14) and (4.15) are inserted into Eq. (4.3), we find

$$\int_0^{2\pi} U_{Nn\beta}(\theta_1, \ldots, \theta_n) d\theta_n = (N+1-n)\, U_{N,n-1,\beta}(\theta_1, \ldots, \theta_{n-1}), \quad (4.16)$$

which holds for $n = 1, 2, \ldots, N$ if we make the convention

$$U_{N0\beta} = 1. \quad (4.17)$$

We now go back to Eq. (3.44) and integrate both sides with respect to $(\theta_{n+1}, \ldots, \theta_N)$. Taking account of Eq. (1.2), (1.8) and (4.16), we find for $n = 0, 1, \ldots, N$,

$$U_{Nn\beta}(\theta_1, \ldots, \theta_n) = \eta_{N\beta} R_{Nn\beta}(\theta_1, \ldots, \theta_n). \quad (4.18)$$

Taking $n = 0$ and using Eq. (1.9) and (4.17), we obtain

$$\eta_{N\beta} = 1, \quad (4.19)$$

and the proof of Theorems 3 and 4 is complete.

V. Mathematical Note

The results of this paper are based upon the use of the Circular Ensembles [1, 2] which are better known to mathematicians by the name of Symmetric Spaces. The Circular Ensemble $E_\beta(N)$ is the Symmetric Space

$$[U(N)/O(N)], \quad \beta = 1, \quad (5.1)$$

$$U(N), \quad \beta = 2, \quad (5.2)$$

$$[U(2N)/\mathrm{Sp}(2N)], \quad \beta = 4, \quad (5.3)$$

with a probability-distribution which is defined, by the invariant measure, to be uniform on the entire space. The points of the space $E_\beta(N)$ are unitary matrices having N eigenvalues

$$e^{i\theta_1}, \ldots, e^{i\theta_N}. \quad (5.4)$$

It is these eigenvalues which have the joint probability-distribution defined by Eq. (1.2) and the correlation-functions specified by Theorem 4.

The proof of Theorem 4 in this paper is a mere verification. It would be highly desirable to find a more illuminating proof, in which the appearance of the quaternion-determinant (3.19) might be related directly to the structure of the symmetric space $E_\beta(N)$.

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