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Vicious Walkers and Hook Young Tableaux

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ABSTRACT

We consider a generalization of the vicious walker model. Using a bijection map between the path configuration of the non-intersecting random walkers and the hook Young diagram, we compute the probability concerning the number of walker's movements. Applying the saddle point method, we reveal that the scaling limit gives the Tracy–Widom distribution, which is same with the limit distribution of the largest eigenvalues of the Gaussian unitary ensemble.

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1 Introduction

Since it was shown that the path configuration of the random vicious walkers [1] is related with the Young tableaux [2–4], much attention has been paid on the statistical combinatorial problems which are intimately related with the Young tableaux. Among them is the random permutation [5], the random word [6], the point process [7,8], the random growth model (the polynuclear growth model, oriented digital boiling model) [9,10], the queuing theory [11], and so on. Interesting is that the scaling limits of these models have the universality that the fluctuation is of order $N^{1/3}$ with the mean being of order N , and that the asymptotic distribution of appropriately scaled variables is described by the Tracy–Widom distribution, which was originally identified with the limit distribution for the largest eigenvalue of the Gaussian unitary random matrix [12]. See Refs. 13–16 for a review.

In this paper motivated from results in Ref. 17 and conjectures in Ref. 18, we introduce a physical model of the vicious walkers based on the hook Young tableaux. We shall study the scaling limit of certain probability, and clarify a relationship with the Tracy–Widom distribution.

For our later convention we define the (M, N) -hook Schur functions (or, sometimes called the supersymmetric Schur function) [19] (see also Refs. 20–22), and denote some properties of the hook Young tableaux briefly. We set $\mathbf{B} = \mathbf{B}_+ \sqcup \mathbf{B}_-$, and

$$\mathbf{B}_+ = \{\epsilon_1, \dots, \epsilon_M\}, \quad \mathbf{B}_- = \{\epsilon_{M+1}, \dots, \epsilon_{M+N}\}. \quad (1.1)$$

Hereafter we call i as positive (resp. negative) symbol when $\epsilon_i \in \mathbf{B}_+$ (resp. $\epsilon_i \in \mathbf{B}_-$). We fix an ordering in \mathbf{B} as

$$\epsilon_1 < \epsilon_2 < \dots < \epsilon_{M+N}. \quad (1.2)$$

It should be noted that, though we use an ordering (1.2), following discussion can be applied for any other choices of ordering with $|\mathbf{B}_+| = M$ and $|\mathbf{B}_-| = N$. For a given Young diagram λ , the semi-standard Young tableaux (SSYT) T is given by filling a number $1, 2, \dots, M + N$ in λ by the following rules;

- the entries in each row are increasing, allowing the repetition of positive symbols, but not permitting the repetition of negative symbols,

- the entries in each column are increasing, allowing the repetition of negative symbols, but not permitting the repetition of positive symbols.

We define the weight for SSYT T as

$$\text{wt}(T) = \sum_{a=1}^{M+N} m_a \epsilon_a, \quad (1.3)$$

where m_a is the number of a 's in T . Then the hook Schur function $S_\lambda(x, y)$ is given by

$$S_\lambda(x, y) = \sum_{\text{SSYT } T \text{ of shape } \lambda} e^{\text{wt}(T)}. \quad (1.4)$$

Here we have used

$$\begin{cases} x_i = e^{\epsilon_i}, & \text{for } \epsilon_i \in \mathbf{B}_+, \\ y_j = e^{\epsilon_{M+j}}, & \text{for } \epsilon_{M+j} \in \mathbf{B}_-. \end{cases}$$

The Schur function $s_\lambda(x)$ in usual sense corresponds to a case of $\mathbf{B}_- = \emptyset$ ($N = 0$), and the hook Schur function S_λ with $\mathbf{B}_+ = \emptyset$ ($M = 0$) reduces to the Schur function for the conjugate partition λ' ;

$$S_\lambda(x, 0) = s_\lambda(x), \quad S_\lambda(0, y) = s_{\lambda'}(y). \quad (1.5)$$

Due to the rule of filling a number, the Young diagram λ should be contained in the (M, N) -hook (see Fig. 1), and we have

$$S_\lambda(x, y) = \sum_{\mu \subset \lambda} s_\mu(x) s_{\lambda'/\mu'}(y).$$

Furthermore when λ contains the partition (N^M) , we have

$$S_\lambda(x, y) = s_\mu(x) s_\nu(y) \prod_{i=1}^M \prod_{j=1}^N (x_i + y_j),$$

where the partitions μ and ν are defined from λ by $\mu_i = \lambda_i - N$ and $\nu_j = \lambda'_j - M$, respectively.

The Jacobi–Trudi formula helps us to write the hook Schur function as

$$S_\lambda(x, y) = \det \left(c_{\lambda_i + j - i} \right)_{1 \leq i, j \leq \ell(\lambda)}, \quad (1.6)$$

Here $\ell(\lambda)$ is the length of λ , and c_n is given by the generating function,

$$H(t; x) E(t; y) = \sum_{n=0}^{\infty} c_n t^n, \quad (1.7)$$

with

$$H(t; x) = \prod_j \frac{1}{1 - t x_j}, \quad E(t; y) = \prod_j (1 + t y_j). \quad (1.8)$$

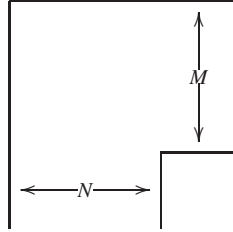


Figure 1: (M, N) -hook Young diagram must be contained in above “hook” region.

This paper is organized as follows. In section 2 we introduce a model of vicious walkers as a generalization of the original model [1]. As far as we know, this model is first presented in this paper. We define the bijection from path configurations of vicious walker to the hook Young diagram. Especially we show a relationship between the length of the Young diagram and the number of movements of the first walker. This type of bijection was first given in Refs. 2, 3 for the original vicious walker model. In section 3 we give the probability of $\ell(\lambda) \leq \ell$ in terms of the Toeplitz determinant. We further study the scaling limit of this probability based on the transformation identity from the Toeplitz determinant to the Fredholm determinant [23–25] in section 4. We apply the saddle point method to the Fredholm determinant following Refs. 10, 26, and show that the scaling limit coincides with the Tracy–Widom distribution for the GUE [12]. In section 5 we consider some simple examples as a reduction of our model. Both the Meixner and the Krawtchouk ensembles can be regarded as a reduction of our vicious walker model. The last section is for conclusion and discussions. We briefly comment on the random word related with the hook Young tableaux.

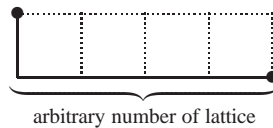
2 Vicious Walker

We define a model of the random walkers which is related with the hook Schur function (1.4). The model is a generalization of one introduced in Ref. 1, and as will be clarified later an algebraic property of the partition function is nothing but an identity in Ref. 17.

Evolution rule of vicious walkers is defined as follows. Initially there are infinitely many walkers at $\{\dots, -2, -1\}$, and we call each walker P_j whose initial point is $-j$. A walker is movable rightward if its right site is vacant. Walkers P_{j+1}, P_{j+2}, \dots are called successors of a walker P_j if they are next to each other in the order of the indices. We consider two types of time evolution (we assume that there are totally $M + N$ time steps); first M -steps are referred as “normal” time evolution, and following N -steps are as “super” time evolution. At a “normal” time evolution, a movable walker either stays or moves to its right together with an arbitrary number of its successors. Thus we draw



On the other hand, at a “super” time evolution, a walker can move to its right any number of lattice units, though P_j cannot over-pass a position of P_{j-1} at previous time. To realize this rule and to draw a non-intersecting path, it is convenient to depict this step as follows;



Each path of the vicious walkers is required not to intersect each other. We see that the original model of vicious walkers [1] corresponds to a case of $N = 0$. After $M + N$ -time steps, we denote $L_j(n)$ as the number of right moves made by the walker P_j . Here n is a total number of movements of walkers. In Fig. 2 we give an example of path configuration of vicious walkers. In this case we consider totally 5-time steps ($M = 3$ and $N = 2$), and the total step of right movements is $n = 12$ with $(L_1, L_2, L_3, L_4) = (5, 4, 2, 1)$.

It is now well known for the model of the original vicious walks [1] that we have the bijection from the path configuration of vicious walkers to the Young diagram [2]. This bijection can be easily generalized to our model as follows. For a path configuration (see, *e.g.*, Fig. 2), we draw Young tableaux $\lambda \vdash n$ with $\lambda'_j = L_j(n)$. We insert in the j -th column from top the times at which the j -th particle made a movement to its right. Notice that, for a super time evolution, we prepare number of time as many as lattice units the walker moved. For instance, in a case of Fig. 2, we put “1 2 4 4 5” in the first column, as P_1 moves 2 lattice units rightward at time-4. Thus the top row is the list of times at which the walkers made their first movement, the second row is the list

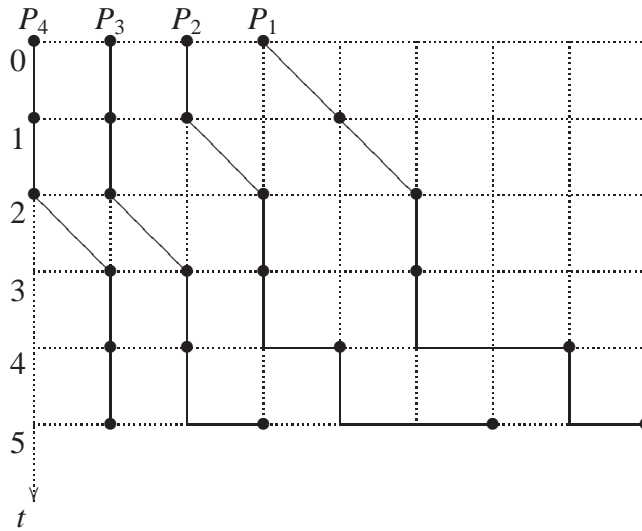


Figure 2: Example of path configuration. For $t \leq 3$, a process is a “normal” evolution, while for $t \geq 4$ it becomes a super time evolution.

of times at which the walkers made their second movement, and so on. It is clear that the normal time corresponds to the positive symbol \mathbf{B}_+ while the super time denotes the negative symbol \mathbf{B}_- in SSYT. The evolution rule supports a consistency with ordering (1.2) in \mathbf{B} , and we know that the map is indeed the bijection. Following this mapping the path configuration in Fig. 2 is mapped to SSYT given in Fig. 3.

1	2	3	3
2	4	5	
4	5		
4	5		
5			

Figure 3: Semi-standard (hook) Young tableaux with entries from $\mathbf{B}_+ = \{1, 2, 3\}$ and $\mathbf{B}_- = \{4, 5\}$.

To summarize, we have a one-to-one correspondence between the path configuration and SSYT; when n is the number of total moves of vicious walkers, we have $\lambda \vdash n$, and the number $L_j(n)$ of right movements made by the j -th walker is equal to the number of boxes in the j -th column of SSYT. Especially the length $\ell(\lambda)$ of partition coincides with $L_1(n)$.

3 Partition Function and Toeplitz Determinant

In the following we consider a model where, after totally n -step movements of the right movers, every walkers return to their initial positions by totally n -step left movements [3]. Here the number of normal (resp. super) time evolution is supposed to be M_1 (resp. N_1) in the first right moves, while the number of normal (resp. super) time evolution is M_2 (resp. N_2) in the next left moves returning to their initial positions. The definition of normal and super time evolution in the left movers simply follows from that of right movers as a mirror image. Applying the bijection in previous section, the path configuration is denoted by pairs of SSYT $\lambda \vdash n$, one is (M_1, N_1) -hook Young tableaux and another is (M_2, N_2) -hook tableaux.

We denote $d_\lambda(M, N)$ as the number of semi-standard Young tableaux of shape λ with entries from $\mathbf{B}_+ \sqcup \mathbf{B}_-$ (with $|\mathbf{B}_+| = M$ and $|\mathbf{B}_-| = N$). By definition, we have $S_\lambda(\underbrace{t, \dots, t}_M, \underbrace{t, \dots, t}_N) = d_\lambda(M, N) t^n$ for $\lambda \vdash n$, and once the Young diagram λ is fixed the number of SSYT $d_\lambda(M, N)$ corresponds to the number of path configuration with fixed end points of right-moves.

We have interests in the probability that the number of right movements of the first walker P_1 is less than ℓ ,

$$\text{Prob}(L_1 \leq \ell). \quad (3.1)$$

Here the probability ‘‘Prob’’ is defined as follows; we assign the weight t (we set $0 < t < 1$) for every right- and left-moves, and regard the weight of totally n -step walk as t^n . Then each configuration of random walk, in which every walkers return to their initial positions after total $2n$ -step, is realized with a probability t^{2n}/Z . An explicit form of the normalization factor Z will be given later. Based on the bijection map studied in previous section, we find that the probability (3.1) is given explicitly by

$$\text{Prob}(L_1 \leq \ell) = \frac{1}{Z} \sum_n \left(\sum_{\substack{\lambda \vdash n \\ \ell(\lambda) \leq \ell}} d_\lambda(M_1, N_1) d_\lambda(M_2, N_2) \right) t^{2n}. \quad (3.2)$$

Note that a normalization factor is set to be $\lim_{\ell \rightarrow \infty} \text{Prob}(L_1 \leq \ell) = 1$.

To relate this probability with the random matrix theory, we follow a method in Ref. 6.

Applying the Gessel formula to eq. (1.6), we have

$$\begin{aligned}
& \sum_{\ell(\lambda) \leq \ell} S_\lambda(x, y) S_\lambda(z, w) \\
&= \frac{1}{\ell! (2\pi)^\ell} \int_{-\pi}^{\pi} d\theta \prod_{1 \leq j < k \leq \ell} |e^{i\theta_j} - e^{i\theta_k}|^2 \prod_{j=1}^{\ell} H(e^{i\theta_j}; x) E(e^{i\theta_j}; y) H(e^{-i\theta_j}; z) E(e^{-i\theta_j}; w) \\
&= D_\ell(\varphi),
\end{aligned} \tag{3.3}$$

where $\varphi(z)$ is defined by

$$\varphi(z) = \prod_{i,j} \frac{1 + y_i z^{-1}}{1 - x_j z^{-1}} \frac{1 + w_i z}{1 - z_j z}. \tag{3.4}$$

We have used $D_\ell(\varphi)$ as the Toeplitz determinant for the function $\varphi(z)$; $D_\ell(\varphi)$ is the determinant of $\ell \times \ell$ matrix where an (i, j) -element is given by φ_{i-j} with $\varphi(z) = \sum_{n \in \mathbb{Z}} \varphi_n z^n$. We note that eq. (3.3) was also given in Ref. 17. Thus our model of random walkers corresponds to a point process in Ref. 17 which was introduced as a generalization of Ref. 7. We note that the strong Szegő limit theorem gives a generalization of the Cauchy formula,

$$\lim_{\ell \rightarrow \infty} D_\ell(\varphi) = \prod_{i,j,m,n} \frac{(1 + x_i w_n)(1 + y_j z_n)}{(1 - y_j w_n)(1 - x_i z_m)}. \tag{3.5}$$

We now apply a principal specialization ps which set $x_i = a q^i$ and $y_j = b q^j$ [22]. In general we have

$$\text{ps}(S_\lambda(x, y)) = S_\lambda(\underbrace{a q, a q^2, \dots}_{\infty}, \underbrace{b q, b q^2, \dots}_{\infty}) = q^{\sum_{i=1}^{\ell(\lambda)} i \lambda_i} \prod_{(i,j) \in \lambda} \frac{a + b q^{j-i}}{1 - q^{\lambda_i - j + \lambda'_j - i + 1}},$$

and for a case of $\lambda \vdash n$ and (M, N) -hook Young diagram, by definition we have by setting $a = b = t$ and $q = 1$

$$\text{ps}_{a=b=t, q=1}(S_\lambda(x_1, \dots, x_M, y_1, \dots, y_N)) = d_\lambda(M, N) t^n.$$

As a result, from eq. (3.3) we obtain the partition function as

$$\sum_n \sum_{\substack{\ell(\lambda) \leq \ell \\ \lambda \vdash n}} d_\lambda(M_1, N_1) d_\lambda(M_2, N_2) t^{2n} = D_\ell(\tilde{\varphi}), \tag{3.6}$$

where

$$\tilde{\varphi}(z) = \frac{(1 + t z^{-1})^{N_1} (1 + t z)^{N_2}}{(1 - t z^{-1})^{M_1} (1 - t z)^{M_2}}. \tag{3.7}$$

Due to the strong Szegö limit theorem, we obtain a normalization factor Z as

$$Z = \lim_{\ell \rightarrow \infty} D_\ell(\tilde{\varphi}) = \frac{(1+t^2)^{M_1 N_2 + M_2 N_1}}{(1-t^2)^{M_1 M_2 + N_1 N_2}}. \quad (3.8)$$

Combining these results, we get

$$\text{Prob}(L_1 \leq \ell) = \frac{1}{Z} D_\ell(\tilde{\varphi}). \quad (3.9)$$

4 Scaling Limit

We study the asymptotic behavior of the probability (3.1). We note that in Ref. 18 the property of the scaling limit was conjectured. For our purpose, it is generally useful to rewrite the Toeplitz determinant with the Fredholm determinant. In fact, once we know the Toeplitz determinant, it is possible to rewrite it in terms of the Fredholm determinant [23–25]. Namely we have

$$D_\ell(\tilde{\varphi}) = Z \det(1 - \mathcal{K}_\ell), \quad (4.1)$$

where Z is defined in eq. (3.8), and \mathcal{K}_ℓ is the matrix defined by

$$\mathcal{K}_\ell(i, j) = \sum_{k=0}^{\infty} (\tilde{\varphi}_- / \tilde{\varphi}_+)_{i+\ell+k+1} (\tilde{\varphi}_+ / \tilde{\varphi}_-)_{-j-\ell-k-1}. \quad (4.2)$$

Here a subscript denotes the Fourier component of the function, and we have used the Wiener–Hopf factor of $\tilde{\varphi}$, $\tilde{\varphi} = \tilde{\varphi}_+ \tilde{\varphi}_-$,

$$\tilde{\varphi}_+ = \frac{(1+tz)^{N_2}}{(1-tz)^{M_2}}, \quad \tilde{\varphi}_- = \frac{(1+tz^{-1})^{N_1}}{(1-tz^{-1})^{M_1}}.$$

Note that we have set $0 < t < 1$. The probability (3.1) is now written by the Fredholm determinant as

$$\text{Prob}(L_1 \leq \ell) = \det(1 - \mathcal{K}_\ell). \quad (4.3)$$

Using a representation (4.3) in terms of the Fredholm determinant, we study an asymptotic behavior by applying the saddle point method following Refs. 10, 26. We consider a limit $M_a, N_a \rightarrow \infty$ for $a = 1, 2$ with fixed values;

$$\frac{M_1}{N_2} = m_1, \quad \frac{N_1}{N_2} = n_1, \quad \frac{M_2}{N_2} = m_2.$$

In the Fredholm determinant (4.2), matrix elements are computed as

$$\begin{aligned} (\tilde{\varphi}_+/\tilde{\varphi}_-)_{-\ell-j-k-1} &= \oint \frac{dz}{2\pi i} \frac{(1+tz)^{N_2}}{(1-tz)^{M_2}} \cdot \frac{(1-t/z)^{M_1}}{(1+t/z)^{N_1}} z^{j+k+\ell}, \\ (\tilde{\varphi}_-/\tilde{\varphi}_+)_{\ell+i+k+1} &= \oint \frac{dz}{2\pi i} \frac{(1-tz)^{M_2}}{(1+tz)^{N_2}} \cdot \frac{(1+t/z)^{N_1}}{(1-t/z)^{M_1}} z^{-i-k-\ell-2}. \end{aligned}$$

A path of integration in the former integral is chosen in a way that it surrounds $z = -t$, and that $z = 1/t$ is outside. On the other hand, a path of the latter integral includes both $z = 0$ and $z = t$ while it excludes $z = -1/t$. We set

$$\ell = c N_2 + s N_2^{\frac{1}{3}}, \quad (4.4)$$

where c is to be fixed later. For brevity, we define the function $\sigma(z)$ by

$$\sigma(z) = m_1 \log(t-z) - n_1 \log(t+z) + \log(1+tz) - m_2 \log(1-tz) + (-m_1 + n_1 + c) \log z. \quad (4.5)$$

Then above integrals are given by

$$\begin{aligned} (\tilde{\varphi}_+/\tilde{\varphi}_-)_{-\ell-j-k-1} &= (-1)^{M_1} \oint \frac{dz}{2\pi i} e^{N_2 \sigma(z)} z^{j+k+s N_2^{1/3}} \equiv (-1)^{M_1} I_1, \\ (\tilde{\varphi}_-/\tilde{\varphi}_+)_{\ell+i+k+1} &= (-1)^{M_1} \oint \frac{dz}{2\pi i} e^{-N_2 \sigma(z)} z^{-s N_2^{1/3} - i - k - 2} \equiv (-1)^{M_1} I_2. \end{aligned}$$

We scale matrix indices as $(i, j, k) \rightarrow (N_2^{1/3} x, N_2^{1/3} y, N_2^{1/3} w)$, and we consider to apply the saddle point method to integrals,

$$\begin{aligned} I_1 &= \int_{\mathcal{C}^+} \frac{dz}{2\pi i} e^{N_2 \sigma(z)} z^{N_2^{1/3}(w+y+s)}, \\ I_2 &= \int_{\mathcal{C}^-} \frac{dz}{2\pi i} e^{-N_2 \sigma(z)} z^{-N_2^{1/3}(w+x+s)-2}, \end{aligned}$$

in a limit $N_2 \rightarrow \infty$. In these integrals, we fix a parameter c in eq. (4.4) so that we have a *double* saddle point, namely as a solution of a set of equations,

$$\frac{d\sigma(z)}{dz} = \frac{d^2\sigma(z)}{dz^2} = 0,$$

i.e.,

$$\frac{m_1}{1-z/t} + \frac{1}{1+tz} = c - m_2 + 1 + \frac{m_2}{1-tz} + \frac{n_1}{1+z/t}, \quad (4.6a)$$

$$\frac{m_1}{(t-z)^2} - \frac{n_1}{(t+z)^2} = -\frac{c - m_1 + n_1}{z^2} + \frac{m_2 t^2}{(1-tz)^2} - \frac{t^2}{(1+tz)^2}. \quad (4.6b)$$

This set of equations is rewritten as

$$c = \frac{t}{t-z_0} m_1 - \frac{t}{t+z_0} n_1 - \frac{t z_0}{1-t z_0} m_2 - \frac{t z_0}{1+t z_0}, \quad (4.7a)$$

where z_0 satisfies

$$\frac{m_1}{(t-z_0)^2} + \frac{n_1}{(t+z_0)^2} = \frac{1}{(1+t z_0)^2} + \frac{m_2}{(1-t z_0)^2}. \quad (4.7b)$$

We see that eq. (4.7b) always has a real solution in $(-1/t, -t)$ as far as $n_1 \neq 0$ because l.h.s. – r.h.s. of eq. (4.7b) changes from $-\infty$ to ∞ in $z \in (-1/t, -t)$. Generally real solutions of eq. (4.7b) are not only in $(-1/t, -t)$, but to deform paths of integrals adequately we see that $z_0 \in (-1/t, -t)$ is a unique candidate of a *double* saddle point. For example, in a case of $m_1 = m_2$ and $n_1 = 1$, real solutions of eq. (4.7b) are only $z = \pm 1$, and we can conclude that a double saddle point should be $z_0 = -1$ from a discussion below. In a case of $m_1 = n_1 \geq 1$ and $m_2 = 1$, real solutions of eq. (4.7b) are in $(-1/t, -t)$, $(t, 1/t)$, $(-\infty, -1/t)$, and $(1/t, \infty)$ (the last 2 solutions exist only if $m_1 = n_1 \geq 1/t^2$), and from a discussion to deform contours we see that only $z_0 \in (-1/t, -t)$ is possible as a double saddle point. Based on these cases, it may be natural to conclude that we choose $z_0 \in (-1/t, -t)$ as a double saddle point.

Hereafter we set a double saddle point z_0 so that $z_0 \in (-1/t, -t)$, and fix a parameter c by eq. (4.7a). With $z_0 \in (-1/t, -t)$, we find that $c > 0$ from a definition (4.7a). With this choice of parameters, the fourth order equation (4.6a) has a real solution z_0 of multiplicity two, and other 2 solutions are in $(-t, t)$ and $(1/t, \infty)$. Around z_0 , we have a steepest descend path as in Fig. 4. As z_0 is a double saddle point, paths come into z_0 with angles $\pm\pi/3$ and $\pm 2\pi/3$. Following Ref. 10, we denote such paths as \mathcal{C}^+ and \mathcal{C}^- respectively. We see that original paths explained above eq. (4.4) can be deformed smoothly to contours \mathcal{C}^\pm avoiding their singularities. Furthermore, we have

$$\begin{aligned} & \frac{1}{2} \left. \frac{d^3 \sigma(z)}{dz^3} \right|_{z=z_0} \\ &= \frac{t}{z_0^2} \left(\frac{t-2z_0}{(t-z_0)^3} m_1 + \frac{t+2z_0}{(t+z_0)^3} n_1 - \frac{1-2tz_0}{(1-tz_0)^3} m_2 - \frac{1+2tz_0}{(1+tz_0)^3} \right) \\ &= \frac{-z_0}{1-tz_0} \left(\frac{1-t^2}{(t-z_0)^3} m_1 - \frac{1+t^2}{(t+z_0)^3} n_1 + \frac{2t}{(1+tz_0)^3} \right), \end{aligned}$$

where in the first equality we have used eq. (4.7a) to delete a parameter c , and in the second

equality we have erased m_2 using eq. (4.7b). Recalling $z_0 \in (-1/t, -t)$ and $0 < t < 1$, we see that

$$\left. \frac{d^3 \sigma(z)}{dz^3} \right|_{z=z_0} > 0, \quad (4.8)$$

which shows that functions $|e^{\pm N_2 \sigma(z)}|$ have a maximum value at $z = z_0$ on a contour \mathcal{C}^\pm .

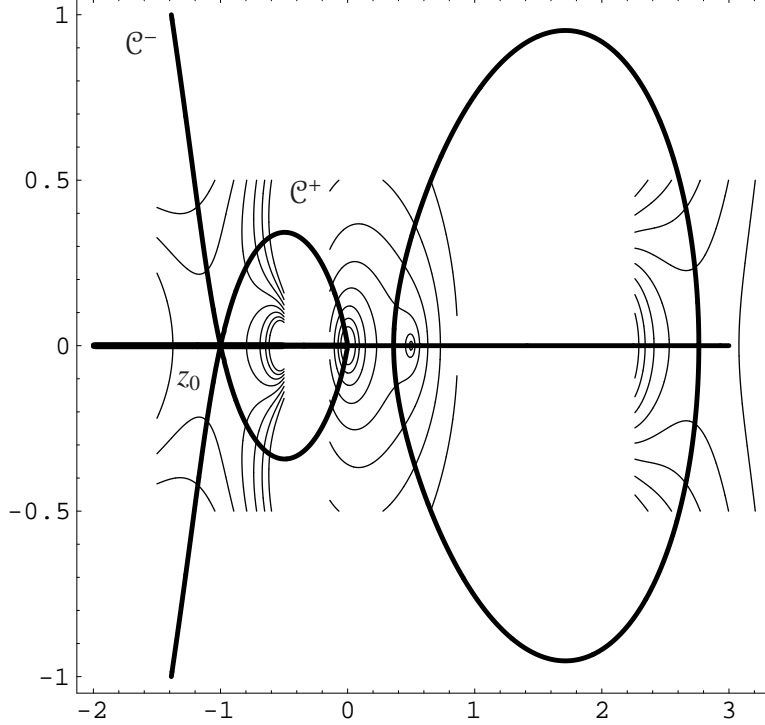


Figure 4: Typical example of the steepest descent path \mathcal{C}^\pm is depicted as a bold line. Here we have set $m_1 = n_1 = m_2 = 1$, and $t = 1/2$. A double saddle point is $z_0 = -1$, and other (simple) saddle points are $(25 \pm 3\sqrt{41})/16$. We see that paths \mathcal{C}^+ and \mathcal{C}^- come to a double saddle point z_0 at angles $\pm\pi/3$ and $\pm 2\pi/3$ respectively, and that another contour comes into (simple) saddle points with angle $\pm\pi/2$. A thin line denotes a local structure of the real part of the integrand around saddle points.

With these settings, we have from the integral I_1 that

$$\begin{aligned} & N_2^{1/3} \int_{\mathcal{C}^+} \frac{dz}{2\pi i} e^{N_2 \sigma(z)} z^{N_2^{1/3}(w+y+s)} \\ &= N_2^{1/3} e^{N_2 \sigma(z_0)} \int_{\mathcal{C}^+} \frac{dz}{2\pi i} e^{\frac{N_2}{6} \sigma'''(z_0)(z-z_0)^3} z^{N_2^{1/3}(w+y+s)} \\ &= N_2^{1/3} z_0^{N_2^{1/3}(w+y+s)} e^{N_2 \sigma(z_0)} \int_{\mathcal{C}^+} \frac{dz}{2i\pi} e^{\frac{N_2}{6} \sigma'''(z_0) z^3 + N_2^{1/3} \frac{w+y+s}{z_0} z} \\ &= -z_0^{N_2^{1/3}(w+y+s)} e^{N_2 \sigma(z_0)} \frac{z_0}{\sigma} \text{Ai}\left(\frac{w+y+s}{\sigma}\right). \end{aligned}$$

Here $Ai(x)$ is the Airy function,

$$Ai(z) = \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{i(zt+t^3/3)},$$

and we have set a parameter σ as

$$\sigma = -z_0 \left(\frac{1}{2} \frac{d^3 \sigma(z)}{dz^3} \Big|_{z=z_0} \right)^{1/3}. \quad (4.9)$$

We have $\sigma > 0$ from eq. (4.8). In the same way, we have from the integral I_2 that

$$N_2^{1/3} \int_{c^-} \frac{dz}{2\pi i} e^{-N_2 \sigma(z)} z^{-N_2^{1/3}(w+x+s)-2} = -z_0^{-N_2^{1/3}(w+x+s)} e^{-N_2 \sigma(z_0)} \frac{1}{z_0 \sigma} Ai\left(\frac{w+x+s}{\sigma}\right).$$

We then see that the kernel of the Fredholm determinant (4.2) is given by the Airy kernel,

$$\frac{1}{\sigma^2} \int_0^{\infty} dw Ai\left(\frac{s+x+w}{\sigma}\right) Ai\left(\frac{s+y+w}{\sigma}\right) = \frac{1}{\sigma} \int_0^{\infty} dw Ai\left(\frac{s+x}{\sigma} + w\right) Ai\left(\frac{s+y}{\sigma} + w\right).$$

As a result, we obtain

$$\lim_{N_2 \rightarrow \infty} \text{Prob} \left(\frac{L_1 - c N_2}{\sigma N_2^{1/3}} \leq s \right) = F_2(s). \quad (4.10)$$

Here $F_2(s)$ is the Tracy–Widom distribution [12] for the scaling limit of the largest eigenvalue of the Gaussian unitary ensemble, and is defined by

$$F_2(s) = \det(1 - \mathcal{K}_{\text{Airy}}) \quad (4.11)$$

$$= \exp \left(- \int_s^{\infty} dx (x-s) q(x)^2 \right). \quad (4.12)$$

The second equality is from Ref. 12, and $q(x)$ is a solution of the Painlevé II equation,

$$q'' = sq + 2q^3, \quad (4.13)$$

with $q(s) \rightarrow Ai(s)$ in $s \rightarrow \infty$.

A proof of convergence would be done along a line of Refs. 10, 26.

To close this section, we comment on a case of $n_1 = 0$. In this case, we further suppose that $m_1/t^2 > 1 + m_2$. With this assumption, we see that there exists a real solution of eq. (4.7b)

in $(-1/t, 0)$. By setting this real solution to be $z_0 \in (-1/t, 0)$, we can prove from eqs. (4.7a) and (4.9) that $c > 0$ and $\sigma > 0$. Note that with this setting of a parameter c eq. (4.6a) has a real solution z_0 of multiplicity two, and another solution is in $(1/t, \infty)$. See Fig. 5 as an example of a steepest descent path. As a result, we obtain the Tracy–Widom distribution (4.10) as a scaling limit.

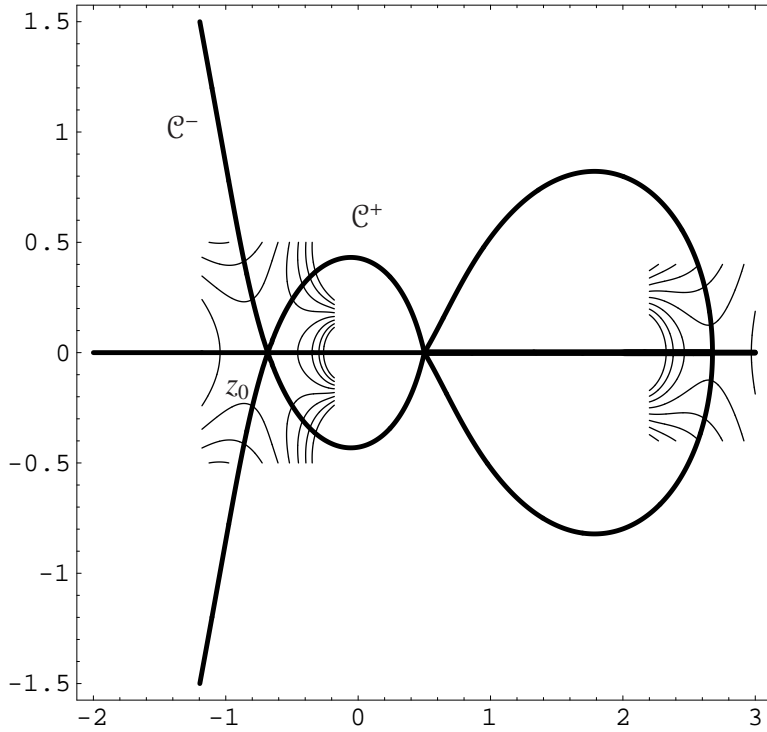


Figure 5: Example of the steepest descent path \mathcal{C}^\pm for a case of $n_1 = 0$ is depicted. Here we have set $m_1 = 4$, $m_2 = 1$, and $t = 1/2$. A double saddle point is $z_0 = -0.68254$, and there is a (simple) saddle point at 2.67684 . As in Fig. 4, paths \mathcal{C}^+ and \mathcal{C}^- come to z_0 at angles $\pm\pi/3$ and $\pm 2\pi/3$ respectively. Another contour comes into a (simple) saddle point with angle $\pm\pi/2$, and it ends at t . A thin line denotes a local structure of the real part of the integrand around saddle points.

5 Some Special Cases

5.1 Meixner Ensemble

We consider a case,

$$M_1 = M_2 = 0, \quad \text{i.e., } m_1 = m_2 = 0.$$

From the viewpoint of the random walkers, the vicious walkers move obeying only super time evolution rule. In this case the Toeplitz determinant (3.6) reduces to

$$\begin{aligned} D_\ell\left(\left(1 + \frac{t}{z}\right)^{N_1} (1 + tz)^{N_2}\right) &= \sum_n \sum_{\substack{\ell(\lambda) \leq \ell \\ \lambda \vdash n}} d_\lambda(N_1) d_\lambda(N_2) t^{2n} \\ &= \sum_n \sum_{\substack{\mu_1 \leq \ell \\ \mu \vdash n}} d_\mu(N_1) d_\mu(N_2) t^{2n}, \end{aligned}$$

where $d_\lambda(N)$ denotes the number of (usual) semi-standard Young tableaux, and we have $d_\lambda(N) = d_\lambda(N, 0) = d_\lambda(0, N)$ from eq. (1.5). The right hand side appeared in Ref. 7, and it gives an example of the discrete orthogonal polynomial ensemble as follows. Using the hook formula [22],

$$d_\mu(M) = \prod_{1 \leq i < j \leq M} \frac{\mu_i - \mu_j + j - i}{j - i}, \quad (5.1)$$

the r.h.s. gives

$$\sum_n \sum_{\substack{\mu_1 \leq \ell \\ \mu \vdash n}} \left(\prod_{1 \leq i < j \leq N_2} \left[\frac{\mu_i - \mu_j + j - i}{j - i} \right]^2 \cdot \left(\prod_{i=1}^{N_2} \left[\prod_{j=N_2-1}^{N_1} \frac{\mu_i + j - i}{j - i} \right] t^{2\mu_i} \right) \right),$$

where we have assumed $N_1 \geq N_2$. Introducing

$$h_j = \mu_j + N_2 - j, \quad (5.2)$$

we obtain

$$\begin{aligned} \text{Prob}(L_1 \leq \ell) &= (1 - t^2)^{N_1 N_2} t^{-N_2(N_2-1)} \left[\prod_{j=0}^{N_2-1} \frac{(N_1 - N_2)!}{j! (N_1 - N_2 + j)!} \right] \\ &\quad \times \sum_{\substack{h \in \mathbb{N}^{N_2} \\ \max\{h_i\} \leq \ell + N_2 - 1}} \left[\prod_{1 \leq i < j \leq N_2} (h_i - h_j)^2 \right] \prod_{i=1}^{N_2} \binom{N_1 - N_2 + h_i}{h_i} t^{2h_i}, \quad (5.3) \end{aligned}$$

which is called the Meixner ensemble.

In fact using the Borodin–Okounkov identity (4.2), the kernel of the Fredholm determinant can be written in terms of the Meixner polynomial

$$\begin{aligned} (i - j) \mathcal{K}(i, j) &= t^{4N_2+i+j} (1 - t^2)^{N_1 - N_2 - 1} \binom{N_1 + j}{N_2 + j} \cdot \binom{N_1}{N_2} \cdot (-N_2) \\ &\quad \times \left(M_{N_2}(i + N_2; N_1 - N_2 + 1, t^2) \cdot M_{N_2-1}(j + N_2; N_1 - N_2 + 1, t^2) \right. \\ &\quad \left. - M_{N_2-1}(i + N_2; N_1 - N_2 + 1, t^2) \cdot M_{N_2}(j + N_2; N_1 - N_2 + 1, t^2) \right), \quad (5.4) \end{aligned}$$

which has a well known form of the correlation functions of the random matrix (see, *e.g.*, Ref. 27). Note that the Meixner polynomial is defined by

$$M_n(x; b, c) = {}_2F_1\left(\begin{matrix} -n, -x \\ b \end{matrix}; 1 - \frac{1}{c}\right).$$

A computation of the scaling limit can be done by the method in Section 4. A double saddle point, $z_0 \in (-1/t, -t)$, is explicitly solved as

$$z_0 = -\frac{t + \sqrt{n_1}}{1 + t \sqrt{n_1}},$$

and we obtain the Tracy–Widom distribution (4.10) with parameters c and σ defined by

$$c = \frac{t(2\sqrt{n_1} + (n_1 + 1)t)}{1 - t^2}, \quad (5.5)$$

$$\sigma = \frac{t^{1/3}(\sqrt{n_1} + t)^{2/3}(1 + t\sqrt{n_1})^{2/3}}{n_1^{1/6}(1 - t^2)}. \quad (5.6)$$

This result was derived by using the asymptotics of the Meixner polynomial in Ref. 7 (see also Ref. 28).

5.2 Krawtchouk Ensemble

We set

$$N_1 = M_2 = 0, \quad \text{i.e., } n_1 = m_2 = 0, \quad (5.7)$$

The vicious walkers obey a rule of normal time evolution in the right moving, while they obey a rule of super time evolution in the left moving. In this case, eq. (3.6) is read as

$$D_\ell \left(\frac{(1 + tz)^{N_2}}{(1 - \frac{t}{z})^{M_1}} \right) = \sum_n \sum_{\substack{\ell(\lambda) \leq \ell \\ \lambda \vdash n}} d_\lambda(M_1) d_{\lambda'}(N_2) t^{2n}.$$

This becomes the Krawtchouk ensemble [29] as follows (this type of the Toeplitz determinant was also studied in Ref. 10). When we substitute the hook formula (5.1) into above expression,

we see that the r.h.s. reduces to

$$\left[\prod_{j=0}^{N_2-1} \frac{(M_1 + j)!}{j!} \right] \sum_n \sum_{\substack{\mu_1 \leq \ell \\ \mu_1 = n}} \left[\prod_{1 \leq i < j \leq N_2} (\mu_i - \mu_j + j - i)^2 \right] \\ \times \prod_{j=1}^{N_2} \frac{t^{2\mu_j}}{(\mu_j + N_2 - j)! (M_1 + j - 1 - \mu_j)!}.$$

By use of

$$h_j = \mu_j + N_2 - j,$$

this gives

$$\text{Prob}(L_1 \leq \ell) = (1 + t^2)^{-M_1 N_2} t^{-N_2(N_2-1)} \left[\prod_{j=0}^{N_2-1} \frac{(M_1 + j)!}{j! (N_2 + M_1 - 1)!} \right] \\ \times \sum_{\substack{h \in \mathbb{N}^{N_2} \\ \max\{h_i\} \leq \ell + N_2 - 1}} \left[\prod_{1 \leq i < j \leq N_2} (h_i - h_j)^2 \right] \prod_{i=1}^{N_2} \binom{M_1 + N_2 - 1}{h_i} t^{2h_i}, \quad (5.8)$$

which is the Krawtchouk ensemble.

The kernel of the Fredholm determinant is computed explicitly from eq. (4.2), and it is given in terms of the Krawtchouk polynomial as a form of the correlation functions;

$$(i - j) \mathcal{K}(i, j) = -\frac{t^{i+j+4N_2}}{(1 + t^2)^{M_1 + N_2}} (M_1 + N_2 - 1) \binom{M_1 + N_2 - 1}{M_1 - 1} \cdot \binom{M_1 + N_2 - 1}{N_2 + j} \\ \times \left(K_{N_2}(i + N_2; \frac{t^2}{1 + t^2}, M_1 + N_2 - 1) \cdot K_{N_2-1}(j + N_2; \frac{t^2}{1 + t^2}, M_1 + N_2 - 1) \right. \\ \left. - K_{N_2-1}(i + N_2; \frac{t^2}{1 + t^2}, M_1 + N_2 - 1) \cdot K_{N_2}(j + N_2; \frac{t^2}{1 + t^2}, M_1 + N_2 - 1) \right). \quad (5.9)$$

Here the Krawtchouk polynomial is defined by

$$K_n(x; p, N) = {}_2F_1 \left(\begin{matrix} -n, -x \\ -N \end{matrix}; \frac{1}{p} \right).$$

We note that we have

$$K_n(x; p, N) = M_n(x; -N, \frac{p}{p-1}).$$

The scaling limit is also computed by the saddle point method [10]. In this case we suppose $m_1 > t^2$, and we have a double saddle point $z_0 \in (-1/t, 0)$ as

$$z_0 = \frac{-\sqrt{m_1} + t}{1 + t \sqrt{m_1}},$$

and we obtain the Tracy–Widom distribution (4.10) where parameters c and σ are defined from eqs. (4.7a) and (4.9) as

$$c = \frac{t(2\sqrt{m_1} + (m_1 - 1)t)}{1 + t^2}, \quad (5.10)$$

$$\sigma = \frac{t^{1/3}(\sqrt{m_1} - t)^{2/3}(1 + t\sqrt{m_1})^{2/3}}{m_1^{1/6}(1 + t^2)}. \quad (5.11)$$

One sees that this result coincides with that of Ref. 29 derived by use of asymptotics of the Krawtchouk polynomial.

5.3 Symmetric Case

We consider a case,

$$M_1 = M_2, \quad N_1 = N_2, \quad (\text{i.e.,} \quad m_1 = m_2 = m, \quad n_1 = 1), \quad (5.12)$$

namely in right- and left-movements we have equal number of normal and super time evolutions. Unfortunately we are not sure whether this model is related with the discrete orthogonal ensemble, but the parameters of the scaling function can be simply solved as follows.

In a scaling limit $N_2 \rightarrow \infty$, we obtain the Tracy–Widom distribution (4.10) by applying the saddle point method. In this case a double saddle point is $z_0 = -1$, and parameters in eq. (4.10) are computed from eqs. (4.7a) and (4.9) as

$$c = \frac{2t(1 + m + (1 - m)t)}{1 - t^2}, \quad (5.13)$$

$$\sigma = \frac{t^{1/3}(m(1 - t)^4 + (1 + t)^4)^{1/3}}{1 - t^2}. \quad (5.14)$$

6 Conclusion and Discussion

We have introduced a generalization of the vicious walker model in Ref. 1. We find that there exists a bijection map between the path configuration of vicious walkers and the hook Young diagram as in the case of the original vicious walkers. We have exactly computed a probability

that the number of right movements of the first walker is less than ℓ , and have given a formula in terms of the Toeplitz determinant. We have further studied a scaling limit of the probability based on the Borodin–Okounkov identity which relates the Toeplitz determinant with the Fredholm determinant, and have obtained the Tracy–Widom distribution for the largest eigenvalue of the Gaussian unitary random matrix. Other models which belong to the orthogonal or the symplectic universality classes are for future studies.

In the case of the vicious walker model, crucial point is that there exists the bijection map from the path configuration to a pair of the semi-standard (hook) Young tableaux. As was well studied [6], a pair of SSYT and the standard tableaux is related with the problem of the random word. We can define the model of the random word which is related with the hook Young diagram as follows [30]. We consider a random word by choosing from a set $\mathbf{B}_+ \sqcup \mathbf{B}_-$ with $\mathbf{B}_+ = \{1, \dots, M\}$ and $\mathbf{B}_- = \{M + 1, \dots, M + N\}$. When a word of length n is given, we have a generalization of the Robinson–Schensted–Knuth (RSK) correspondence [31, 32] (see also Ref. 33 for invariance under ordering of symbols); we have a bijection between a word of length n and pairs (P, Q) of tableaux of the same shape $\lambda \vdash n$ (P is SSYT from \mathbf{B} , and the recording tableaux Q is the standard Young tableaux). Rule to construct pairs of tableaux is essentially same with the original RSK correspondence (see, *e.g.*, Ref. 21, 22), and a difference is only that negative symbols can bump himself while positive symbols cannot bump himself. Then for random word with length n , the probability that the length of longest decreasing (strictly decreasing for positive symbols while weakly decreasing for negative symbols) subsequence is less than or equal to ℓ is then given by

$$\sum_{\substack{\ell(\lambda) \leq \ell \\ \lambda \vdash n}} d_\lambda(M, N) f^\lambda, \quad (6.1)$$

where f^λ is the number of standard Young tableaux.

This can be rewritten in terms of the Toeplitz determinant based on eq. (3.3). We use the exponential specialization [22],

$$\text{ex}(p_n) = t \delta_{1,n}, \quad (6.2)$$

where the power sum symmetric function p_n is given by

$$p_n(x, y) = \sum_i x_i^n + (-1)^{n-1} \sum_j y_j^n.$$

Acting on the hook Schur function, we have

$$\text{ex}(S_\lambda(x, y)) = f^\lambda \frac{t^n}{n!},$$

for $\lambda \vdash n$. By applying the exponential specialization to (x, y) and the principal specialization $\text{ps}_{a=b=t, q=1}$ to (z, w) in eq. (3.3), we get

$$\sum_n \left(\sum_{\substack{\ell(\lambda) \leq \ell \\ \lambda \vdash n}} d_\lambda(M, N) f^\lambda \right) \frac{t^n}{n!} = D_\ell(\Phi), \quad (6.3)$$

where

$$\Phi(z) = e^{t/z} \frac{(1+z)^M}{(1-z)^N}. \quad (6.4)$$

As a consequence, the Poisson generating function of the probability (6.1) is given by the Toeplitz determinant of function Φ . As was seen from the fact that the kernel (6.4) can be given from eq. (3.7) as an appropriate limit, the scaling limit of eq. (6.3) reduces to the Tracy–Widom distribution as was shown in Ref. 16 for a case of $N = 0$. Detail will be discussed elsewhere.

It was shown in Ref. 6 that the generating function (6.3) with $N = 0$ have an integral representation in terms of solutions of Painlevé V equation. It remains for future studies to clarify a relationship between the Toeplitz determinant (6.3) in a case of $N \neq 0$ and the Painlevé equations, especially integral solutions of the Painlevé equation given in Ref. 34.

Note Added: After submitting this paper, Ref. 35 appeared on net. Therein studied was a limit theorem of the “shifted Schur measure”, where the probability is defined in terms of the Schur Q -functions [20]. To apply an a method of Ref. 10 they obtained the Fredholm determinant after a finite perturbation of a product of Hankel operator, but their main result on a scaling limit exactly coincides with our results (4.10) with $M_1 = N_1$ and $M_2 = N_2$ (subsequently one sees that their result for $\tau = 1$ coincides with our above results (5.13)– (5.14) with $m = 1$). This coincidence may originate from a property of the Schur Q -function. The Schur Q -function is defined by filling “marked” and “unmarked” positive integers to the shifted Young diagram; a rule of filling these numbers is much the same with a rule for the semi-standard hook Young tableaux explained in Introduction, once we identify unmarked (resp. marked) numbers with positive (resp. negative) symbols. It will be interesting to investigate this connection in detail.

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