Stability of Equilibrium Points

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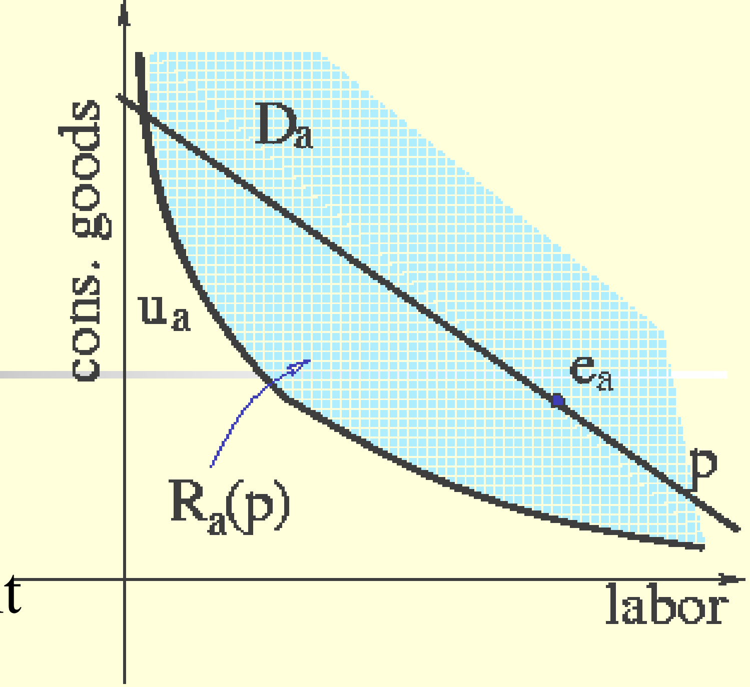
Pure Exchange Economy



Agents: $a \in \mathcal{A}$, $|\mathcal{A}|$ finite

$e_a \in R^n$, goods = endowment

$u_a : R^n \to \overline{R}$ utility function, usc

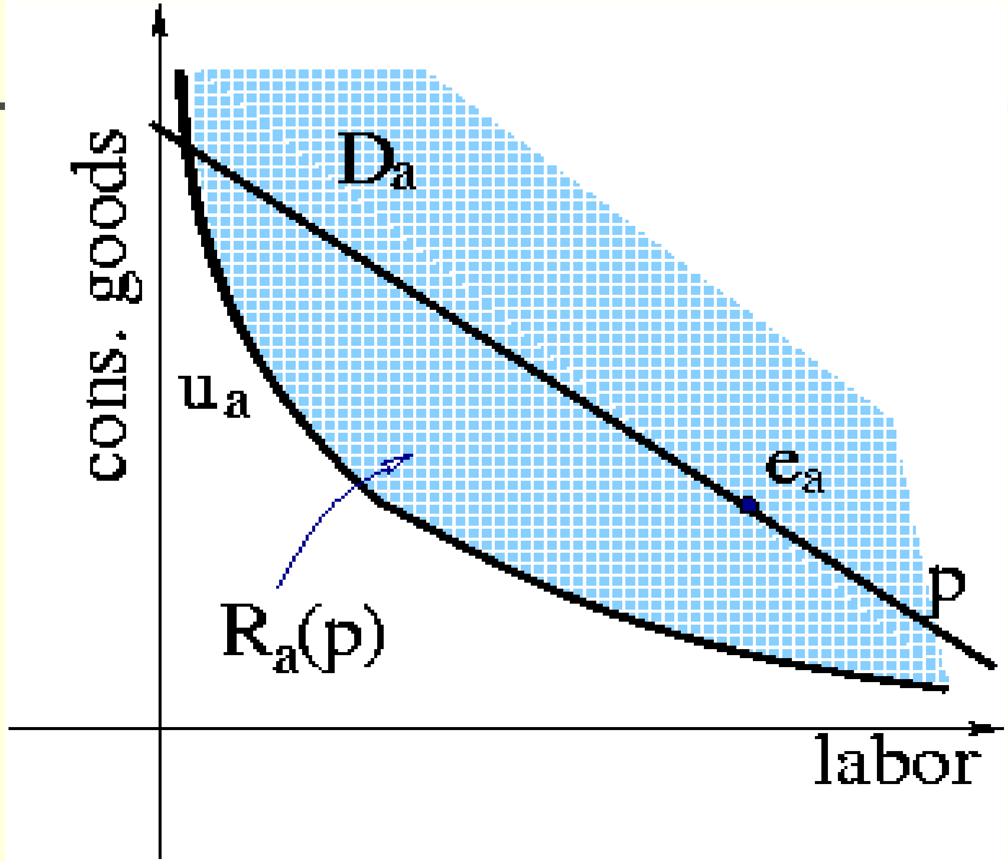
strictly concave (today)

$D_a = \{ c \mid u_a(c) < \infty \} = \text{dom } u_a$ survival set

demand function (of agent *a*)

$$d_a(p) = \operatorname*{argmax}_{c \in R^n} \{ u_a(c) \mid \langle p, c \rangle \le \langle p, e_a \rangle \}$$

Walras equilibrium price



$$\exists\, p \text{ in } \Sigma : s(p) = \Sigma_a e_a - \Sigma_a d_a(p) \geq 0$$

The Walrasian

excess supply : $s(p) = \sum_{a \in A} e_a - \sum_{a \in A} d_a(p)$

$$W(p,q) = \langle q, s(p) \rangle, \quad W : \Sigma \times \Sigma \to R$$

$$d_a(p) = \underset{c \in R^n}{\text{argmax}} \ \{ u_a(c) \mid \langle p, c \rangle \leq \langle p, e_a \rangle \}$$

$p \mapsto d_a(p)$ continuous (hypo-convergence, later)

$\Rightarrow \quad p \mapsto s(p)$ is continuous

Variational Analysis

Optimization Problem: $\max f_0(x)$ so that $x \in C$

Define: $$f(x) = \begin{cases} f_0(x) & \text{if } x \in C \\ -\infty & \text{otherwise} \end{cases}$$

$$\approx \max f(x), \quad x \in \mathbb{R}^n, \quad f : \mathbb{R}^n \to [-\infty, \infty)$$

Lagrangian: $$L(x,y) = f_0(x) + \langle y, G(x) \rangle \text{ if } x \in C,$$
$$= -\infty \text{ if } x \notin C$$

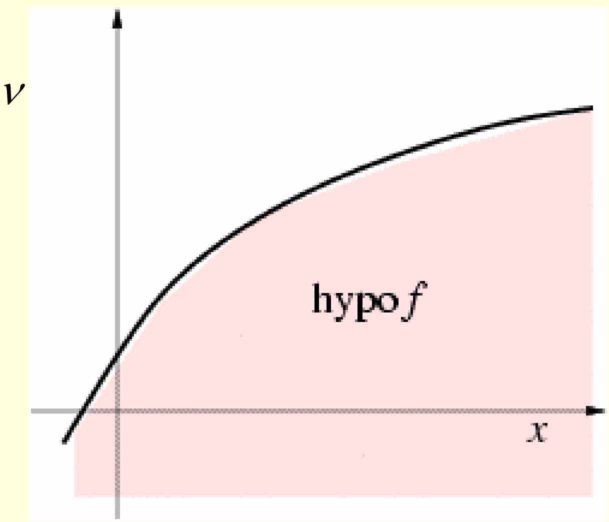
from: $\max f_0(x)$ so that $G(x) = 0, \ x \in C$

Hypo-convergence

- $f^\nu \to_{hypo} f,\ f = \text{hypo} - \lim f^\nu$

$\Downarrow$

- $\text{"}\arg\max f^\nu \to \arg\max f\text{"}$



Hypo-convergence: definition

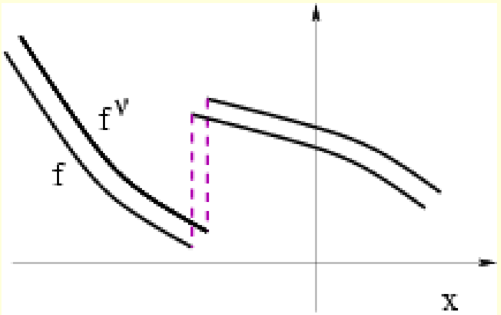
- $\arg\max(f^\nu + g) \to \arg\max(f + g)\ \forall g$ cont.

  requires hypo-convergence
- $f^\nu \to_h f \Leftrightarrow \text{hypo } f^\nu \to \text{hypo } f$
- $\forall x \in \mathbb{R}^n$:

  $(i)\ \ \forall x^\nu \to x, \limsup f^\nu(x^\nu) \le f(x)$

  $(ii)\ \exists x^\nu \to x, \liminf f^\nu(x^\nu) \ge f(x)$

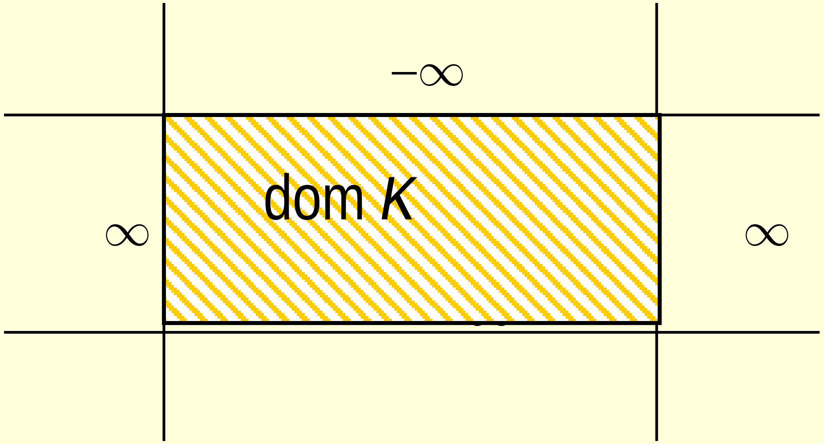
Hypo-convergence: properties



- $f^\nu \to_h f \neq f^\nu \to_p f$
- $f^\nu \to_c f \Rightarrow f^\nu \to_h f$
- $f^\nu \to_h f,\, x^\nu \in \arg\max f^\nu,\, x^{\nu_k} \to \bar{x} \Rightarrow \bar{x} \in \arg\max f$
- $\bar{x} \in \arg\max f \Rightarrow \exists \varepsilon^\nu \searrow 0,\, x^\nu \in \varepsilon^\nu\text{–}\arg\max f^\nu : x^\nu \to \bar{x}$
- $f^\nu \to_h f \Leftrightarrow \text{h-dist}(\text{epi } f^\nu, \text{epi } f) \to 0$

Hypo/epi-convergence

- saddle point $L^a \approx$ saddle point L
- $L : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}},$



- definition: $\forall (x, y)$

$$\text{(a) } \forall x^\nu \to x, \exists\, y^\nu \to y: \quad \limsup L^\nu(x^\nu, y^\nu) \le L(x, y)$$

$$\text{(b) } \forall y^\nu \to y, \exists\, x^\nu \to x: \quad \liminf L^\nu(x^\nu, y^\nu) \ge L(x, y)$$

Lopsided convergence

argmax-inf $K^a \approx$ argmax-inf K

$$\bar{x} \in \arg\max - \inf K \Rightarrow \bar{x} \in \arg\max(\inf_y K(\bullet, y))$$

definition: (Attouch-Wets '83)

$$\text{(a) } \forall x^\nu \to x, \exists y^\nu \to y : \limsup K^\nu(x^\nu, y^\nu) \le K(x,y)$$

$$\text{(b) } \exists x^\nu \to x, \forall y^\nu \to y : \liminf K^\nu(x^\nu, y^\nu) \ge K(x,y)$$

lopsided ➔ hypo-convergence if $K(x,y) = f(x)$,

lopsided ➔ epi-convergence if $K(x,y) = g(y)$

<u>Tight</u> lopsided convergence

definition: $\forall$ *(x,y)*

$$\text{(a) } \forall x^\nu \to x, \exists y^\nu \to y : \limsup K^\nu(x^\nu, y^\nu) \le K(x,y)$$

$$\text{(b) } \exists x^\nu \to x, \forall y^\nu \to y : \liminf K^\nu(x^\nu, y^\nu) \ge K(x,y)$$

tightly:

$$\text{(b-t) } \exists x^\nu \to x, \forall \varepsilon > 0 \;\; \exists \text{ compact } B_\varepsilon : \forall y^\nu \to y :$$

$$\limsup K^\nu(x^\nu, y^\nu) \ge K(x,y)$$

$$\sup_{B_\varepsilon} B^\nu(x^\nu, ..) \ge \sup B^\nu(x^\nu, ..) + \varepsilon, \;\; \forall \nu \ge \nu_\varepsilon$$

Lopsided: basic properties

$$K^\nu \to K \text{ lopsided tightly,}$$

$$\bar{x} \text{ cluster point of } \{x^\nu \in \arg\max-\inf K^\nu, \nu \in \mathbb{N}\}$$

$$\Rightarrow \bar{x} \in \arg\max\text{–}\inf K$$

and

$$\bar{x} \in \arg\max-\inf K \Rightarrow \exists \varepsilon^\nu \to 0,\ \ x^\nu \to \bar{x}$$

$$\text{with } x^\nu \in \varepsilon^\nu\text{–}\arg\max-\inf K^\nu$$

Ky Fan functions & inequality

- $K : \mathbb{R}^n \times \mathbb{R}^n \to \overline{\mathbb{R}}$ is a **Ky Fan fcn** if

  (a) $\forall$ y: $x \mapsto K(x, y)$ usc

  (b) $\forall$ x: $y \mapsto K(x, y)$ convex

- K Ky Fan fcn, dom $K = B \times B$, B compact

  $$\Rightarrow \arg\max - \inf \neq \emptyset$$

  if $K(x, x) \geq 0$ on dom K, $\bar{x} \in \arg\max - \inf K$

  $$\Rightarrow \inf_y K(\bar{x}, y) \geq 0.$$

Walrasian: a Ky Fan fcn!

$$W(p,q) = \langle q, s(p) \rangle$$

(a) $\forall \; q \in \Sigma : \; p \mapsto W(p,q)$ is usc (continuous),

(b) $\forall \; p \in \Sigma : q \mapsto W(p,q)$ is convex (linear),

(c) $\forall \; q \in \Sigma : W(q,q) \geq 0,$ (budget constraint)

$$\forall a : \; \langle q, e_a - d_a(q) \rangle \geq 0$$

$\Rightarrow W$ is a Ky Fan function

Equilibrium price: existence

$W = \langle q, s(p) \rangle$ Ky Fan fcn, Σ compact, convex

$$\Rightarrow \quad \exists\, \bar{p} \in \arg\max_{p \in \Sigma} \left[\inf_{q \in \Sigma} W(p,q) \right]$$

$$\text{and } \inf_{q \in \Sigma} W(\bar{p}, q) \geq 0$$

Claim: $\bar{p}$ is an equilibrium price, i.e., $s(\bar{p}) \geq 0$

$$W(\bar{p}, q) = \langle q, s(\bar{p}) \rangle \geq 0, \quad \forall q \in \Sigma$$

Ky Fan fcns & lopsided

- $K^\nu \to K$ lopsided tightly with $B^\nu \to B$,

  K^ν Ky Fan $\Rightarrow K$ Ky Fan

- and $\forall \nu : \arg\min\text{–}\sup K^\nu \neq \emptyset$

  if $\bar{x} \in$ cluster-pts $\{\arg\max - \inf K^\nu\}$

  $\Rightarrow \bar{x} \in \arg\max - \inf K$ & $K(\bar{x}, \bullet) \geq 0$

- Ky Fan fcns closed under lopsided

  saddle fcns closed under e/h-convergence

  lsc fcns closed under epi-convergence

Brief review

- $W = \langle q, s(p) \rangle$ a Ky Fan fcn, Σ compact
- Ky Fan fcns closed under lopsided tightly
- $W^\nu = \langle q, s^\nu(p) \rangle$: $\arg\min\text{–}\sup W^\nu \neq \emptyset, \forall \nu$
- $\bar{p} \in$ cluster-pts $\{\arg\max - \inf W^\nu\}$

$$\Rightarrow \bar{p} \in \arg\max - \inf W \;\; \& \; W(\bar{p}, \bullet) \geq 0$$

Question: $W^\nu \to_{lop} W$?

tightly guaranteed by Σ compact

Stability issues

- $u_a^\nu \approx u_a$ criterion uncertainty

  $D_a^\nu \approx D_a$ operational uncertainty

  $e_a^\nu \approx e_a$ resources uncertainty

- Convergent economies:
  - $D_a^\nu \to D_a$ (set convergence)
  - $\forall x^\nu \in D_a^\nu \to x \in D_a \Rightarrow u_a^\nu(x^\nu) \to u_a(x)$

    $u_a^\nu \to_c u_a$ "continuous" convergence
  - $e_a^\nu \to e_a$

Equilibrium points: stability

$$u_a^\nu \xrightarrow[c]{} u_a,\ e_a^\nu \to e_a,\ p^\nu \to p \Rightarrow d_a^\nu(p^\nu) \to d_a(p)$$

$$\Rightarrow s^\nu(p^\nu) \to s(p) \quad \text{equiv.} \quad s^\nu \xrightarrow[c]{} s$$

- $\Rightarrow$ lopsided convergence of Walrasians

$$W(p,q) = \langle q, s(p) \rangle, \quad W^\nu(p,q) = \langle q, s^\nu(p) \rangle$$

$$W^\nu \xrightarrow[lop]{} W \qquad \text{Ky Fan fcns closed under lopsided}$$

- i.e. $\arg\max_p \inf_q W^\nu \xrightarrow[\text{cluster}]{} \arg\max_p \inf_q W$

  $\exists\ p^\nu \to p$ (equilibirum points)

Example: Walras Equilibrium

endowments

$$u_2^\nu(c_1, c_2) = c_2 - c_1^{-8}/8 \qquad e_1 = (2, r), e_2 = (r, 2).$$

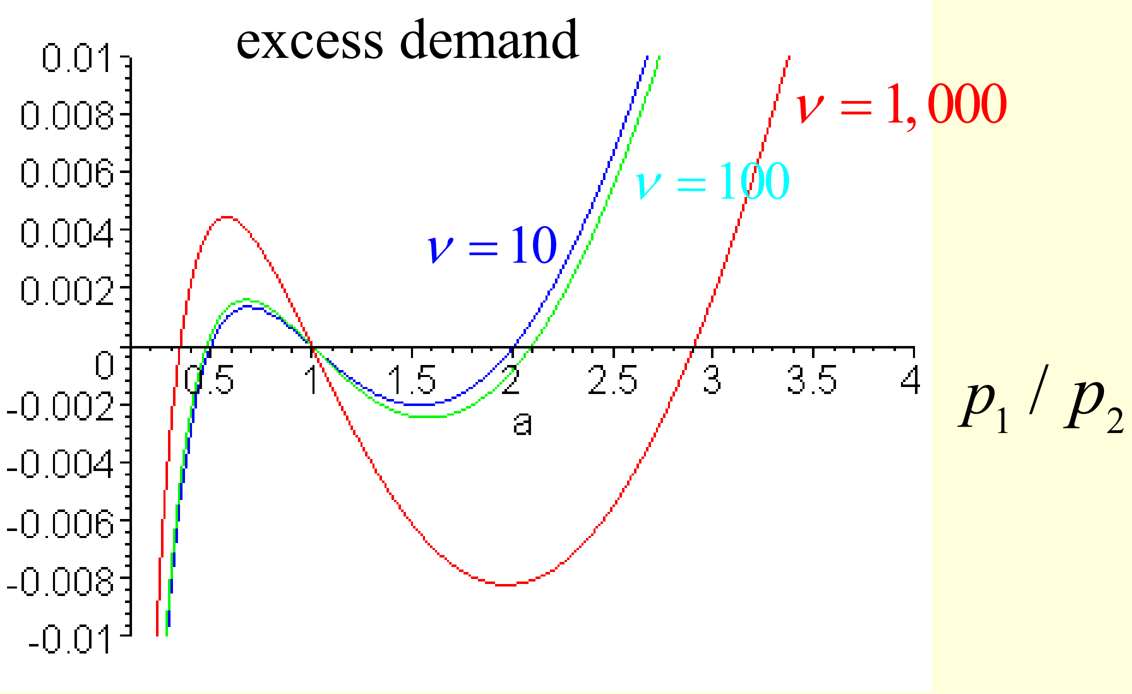
$$u_1^\nu(c_1, c_2) = c_1(1+\nu^{-1}) - c_2^{-8}/8 \text{ (case 1)}$$

$$u_1^\nu(c_1, c_2) = c_1 - c_2^{-8}(1+\nu^{-1})/8 \text{ (case 2)}$$

Case 1: $p_1^\nu / p_2^\nu = 1$ & $p_1^\nu / p_2^\nu \nearrow 0.5$

Case 2: $p_1^\nu / p_2^\nu = 1$ & $p_1^\nu / p_2^\nu \searrow 2$

Equilibrium points: Convergence



$$\text{as } \nu \nearrow \infty : p_1 / p_2 = 0.5, 1, 2$$

Computational implications

- solution procedure via approximation
- use of optimization procedures, or techniques devised for variational inequalities

Augmented Walrasian

$$W(p,q) = \langle q, s(p) \rangle \quad \text{on} \quad \Sigma \times \Sigma$$

$\bar{p}$ max/inf point of W

$$\cong \text{ saddle point } (\bar{p}, \bar{q}) \text{ of } \tilde{W}_r$$

$$\tilde{W}_r(p,q) = \inf_u \left\{ W(p,u) + r\|u\| - \langle q,u \rangle \right\}$$

$$= \sup_z \left\{ W(p,z) \mid \|z-q\|^\circ \le r \right\}$$

with $\|\cdot\|$ a norm and $\|\cdot\|^\circ$ its dual norm

Iterations

$$\tilde{W}_r(p,q) = \sup_z \ \{ \ W(p,z) \ : \ \|z-q\|^\circ \le r \ \}$$

$$q^{k+1} = \underset{q\in\Sigma}{\operatorname{argmax}} \ [\ \max_z \langle z, s(p^k) \rangle \ : \ \|z-q\|^\circ \le r_k \ ]$$

minimizing a linear form on a ball

$$p^{k+1} = \underset{p\in\Sigma}{\operatorname{argmin}} \ [\ \max_z \langle z, s(p^k) \rangle \ : \ \|z-q^{k+1}\|^\circ \le r_{k+1} \ ]$$

reduces to finding the largest element of $s(p^k)$

as $r \uparrow \infty, \ p^k \to \overline{p}$ (max-inf point)

Test case(s)

- Cobb-Douglas utility function:

$$u_a(x) = \gamma_a \prod_{l=1}^{n} x_l^{\beta_l^a} \quad \text{with} \quad \sum_{l=1}^{n} \beta_l^a = 1, \; \beta_l^a \geq 0$$

- budget constraint: $\sum_l p_l x^l \leq \sum_l p_l e_a^l$
- demand (= supply)

$$d_a^l(p) = (\beta_l^a / p_l)\left(\sum_l p_l e_a^l\right), \quad l = 1, \ldots, n$$

experiments: 10 agents, 150 goods (easy!)

More applications

- Fixed point: $\mathbb{B}$ ball, $G:\mathbb{B}\to\mathbb{B}$ continuous

$$\text{find } \overline{x}\in\mathbb{B}: \; G(\overline{x})=\overline{x}, \quad \text{fixed point}$$

- Non-cooperative games: find $\overline{x}$ s.t.

$$\overline{x}_a \in \arg\max u_a(x_a, \overline{x}_{-a}), \; \forall a\in A$$

- Variational Inequality: find $\overline{u}$ s.t.

$$\overline{u}\in C \text{ such that } -G(\overline{u})\in N_C(\overline{u})$$

with C compact convex, G continuous.

Non-cooperative games

- agents: $a \in A, \ |A|$ finite
- $x_a \in C_a \subset R^{n_a}$: decision of agent *a*
- $x_{-a} \in R^{N-n_a}$: decisions of all other agents
- $u_a\left(x_a, x_{-a}\right) : \mathbb{R}^N \to \mathbb{R},$ *a*-performance fcn

Nash equilibrium: $\bar{x} = \left(\bar{x}_a, a \in A\right)$ so that

$$\bar{x}_a \in \arg\max \left\{ u_a(x_a, \bar{x}_{-a}) \mid x_a \in C_a \right\} \quad \forall a \in A$$

The Nash function

$$\bar{x}_a \in \arg\max \tilde{u}_a(x_a, \bar{x}_{-a}), \ \forall a \in A$$

$$\tilde{u}_a(x_a, \bar{x}_{-a}) = u_a(x_a, \bar{x}_{-a}) \text{ unless } x_\alpha \notin C_\alpha \text{ for some } \alpha$$

Nash function: $x = \left(x_a, a \in A\right), y = \left(y_a, a \in A\right)$

$$N(x, y) = \sum_{a \in A} \tilde{u}_a(x_a, x_{-a}) - \sum_{a \in A} \tilde{u}_a(y_a, x_{-a})$$

$\bar{x} = \left(\bar{x}_a, a \in A\right)$ a Nash equilibrium point

$$\Leftrightarrow \bar{x} \in \arg\max_x \inf_y N(x, y)$$

Nash: existence & stability

- Existence: N a Ky Fan function

  $\tilde{u}_a$ usc & $u_a(x_a, ..)$ lsc and $\tilde{u}_a(.., x_{-a})$ concave

  and the sets C_a are bounded (+ closed)

  $\Rightarrow \exists \arg\max - \inf$ point $\bar{x}$ of N

- Stability of equilibrium points $\Leftrightarrow$

  stability of argmax-inf points of N

Stability: sufficient conditions

$$u_a^\nu\left(x_a^\nu, x_{-a}^\nu\right) \to u_a\left(x_a, x_a\right), \forall \left(x_a^\nu, x_{-a}^\nu\right) \to \left(x_a, x_a\right)$$

$$\& \;\; C_a^\nu \to C_a, \;\; C_a \text{ compact}$$

$$\Rightarrow \quad N^\nu \to N \text{ lopsided tightly}$$

$$\Rightarrow \quad \text{Nash equilibrium pts}^\nu \to \text{Nash equilibrium pt}$$