Coping with Uncertainty

in Decision Making Models

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The Newsboy Problem

- $\xi \in \Xi \subset I\!R_+$ demand for a (perishable) good.
- $x \geq 0$ quantity ordered @ unit cost: $c = 10$
- $y \geq 0$ quantity sold, per unit profit $r = 15$

Total revenue (possibly negative):

$$-cx + (c + r)y \text{ where } 0 \leq x, \quad 0 \leq y \leq \min\{x, \xi\}$$

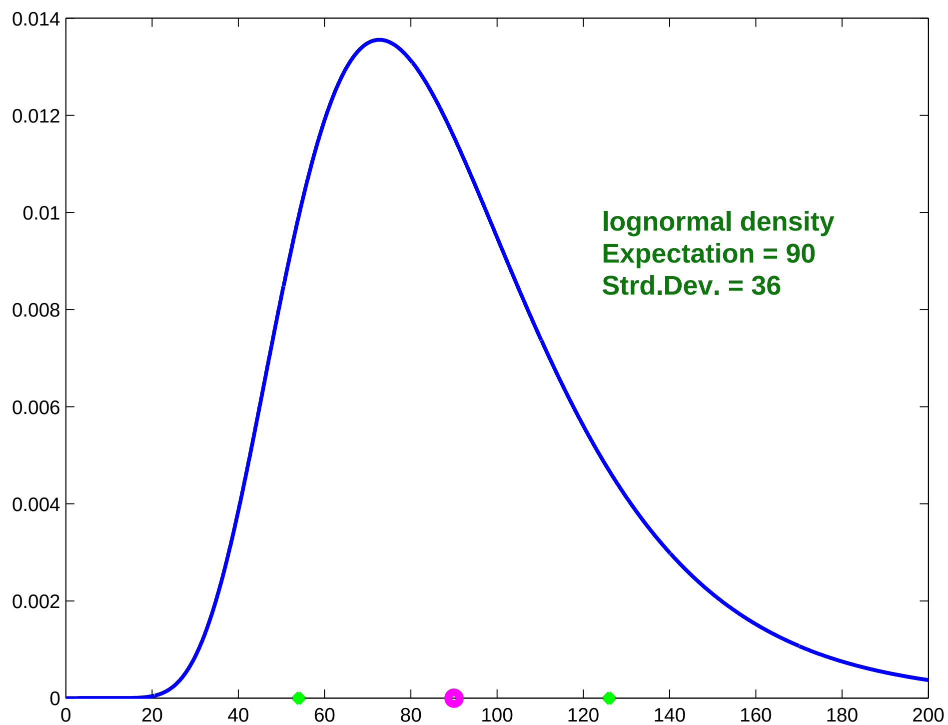
Find optimal x^*!

ξ: Estimated Density h

ξ log-normal: $h(z) = (z\tau\sqrt{2\pi})^{-1}\, e^{-\frac{(\ln z - \theta)^2}{2\tau^2}}$

$\theta = 4.43, \tau = 0.38;\ \ H(z) = \int_0^z h(s)\, ds$

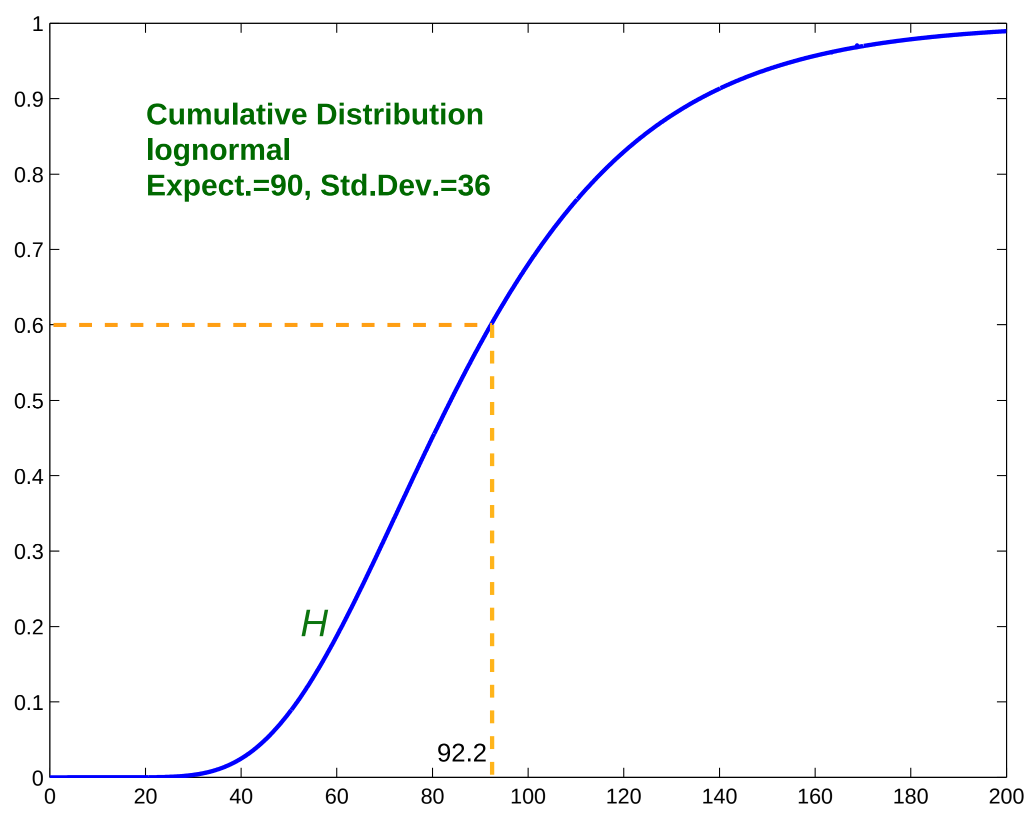
from data, expert(s), all information available



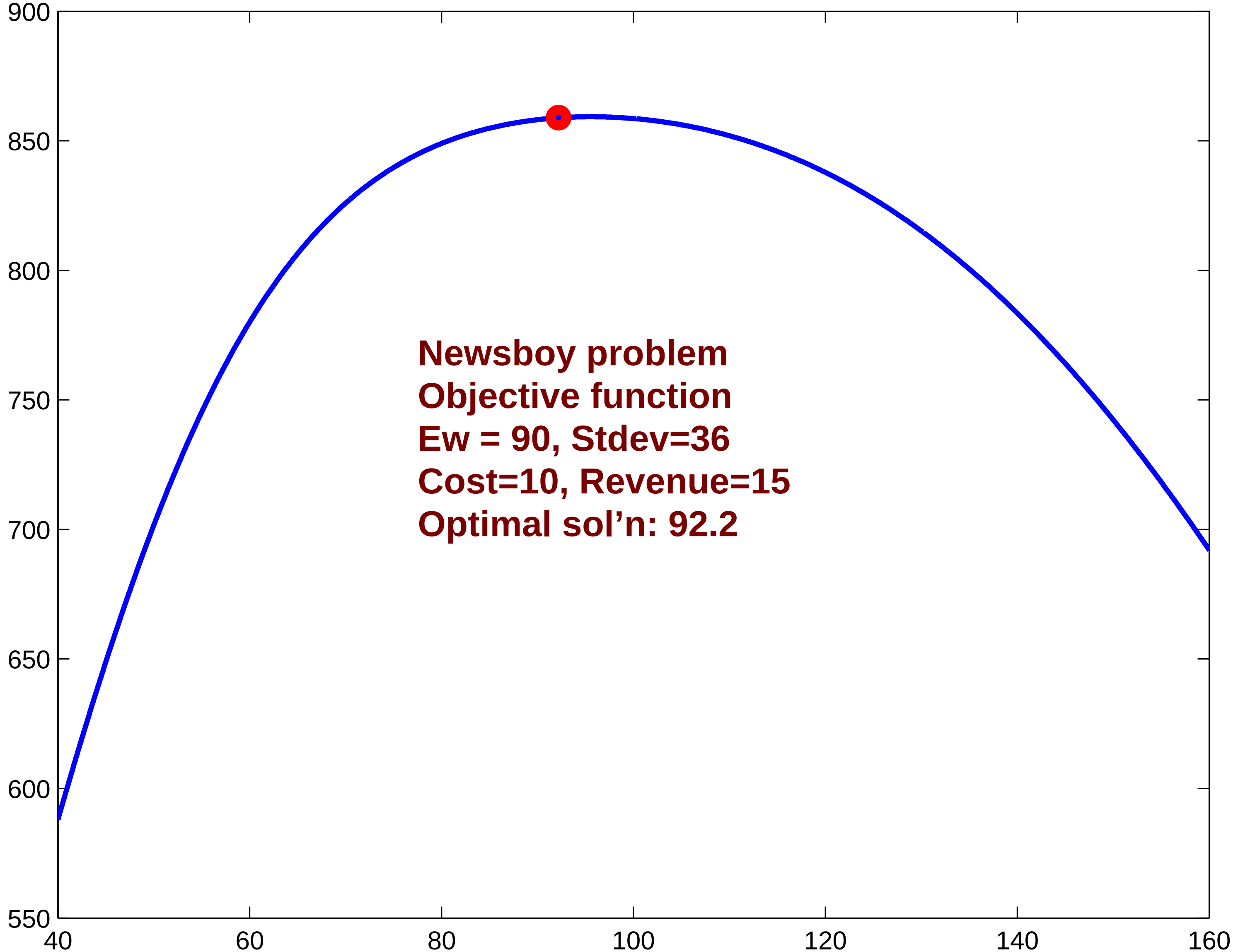
Optimal: Expected Profit

$$x^* = H^{-1}\left(\frac{r}{c+r}\right) = H^{-1}(0.6) = 92.2$$

for $c = 10, r = 15$.

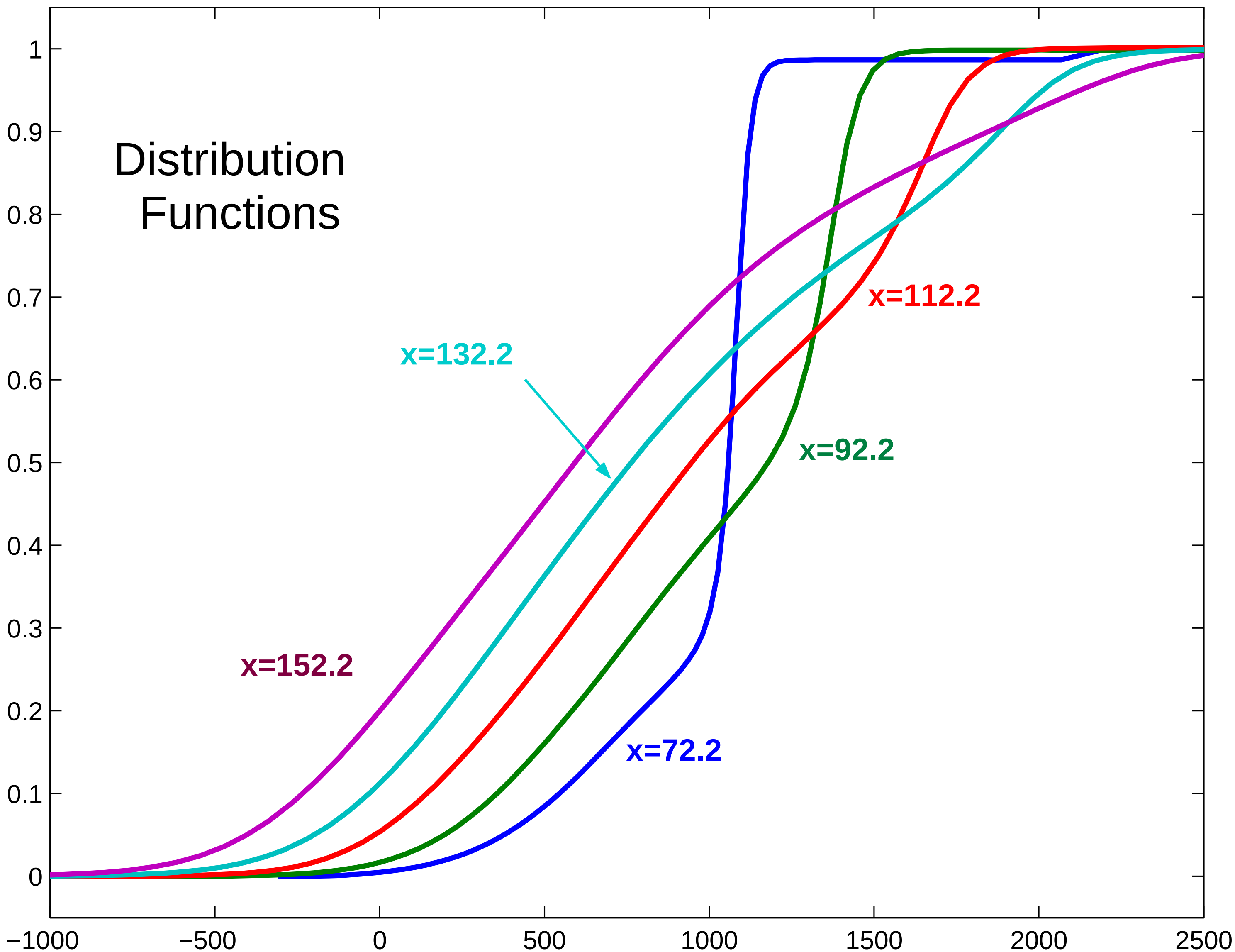


Newsboy’s Objective

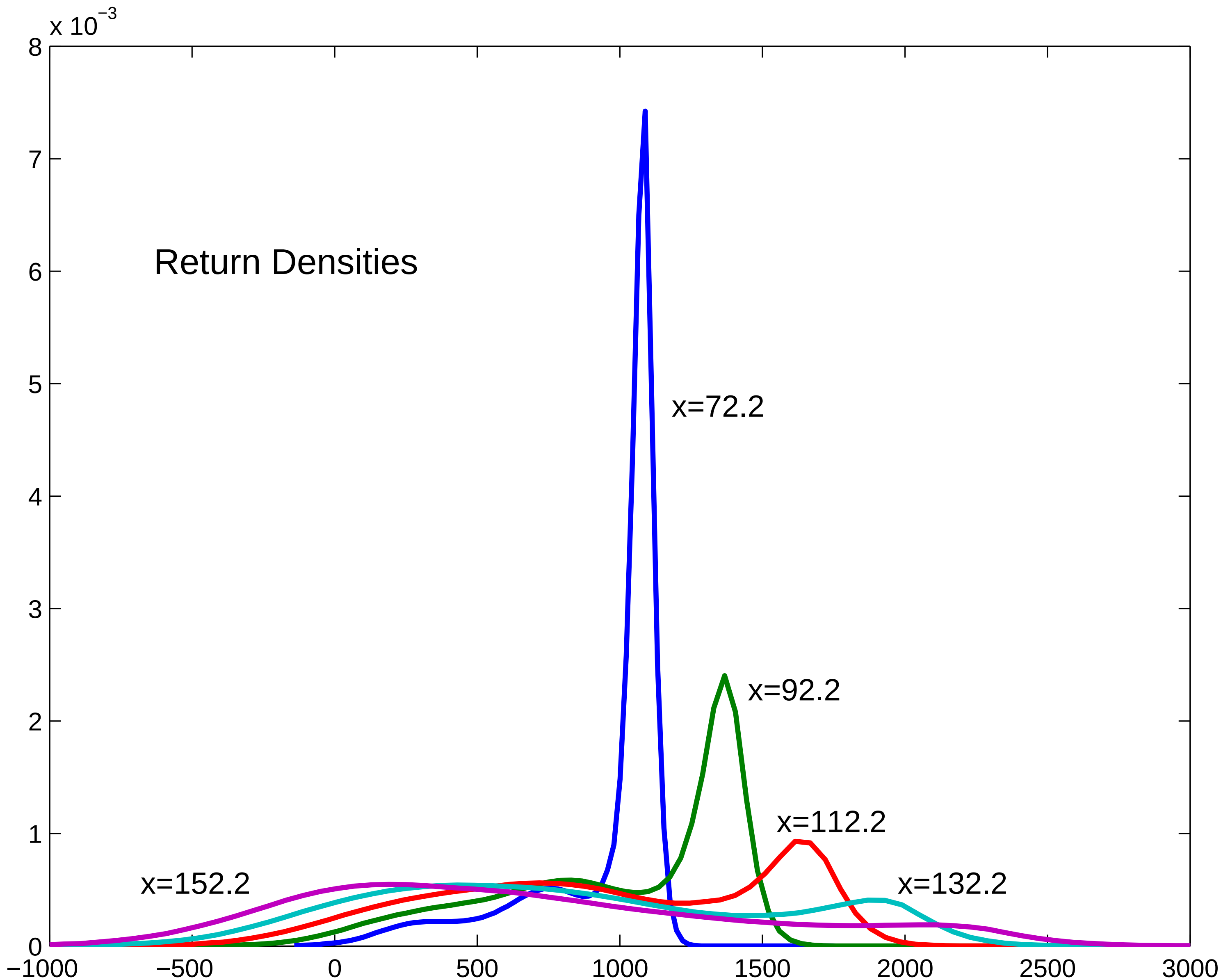


. . . but is maximum expected return the "real" objective?

Choosing the Returns' Distribution



The Returns' Densities



Decision Criteria

Reducing the choice of a distribution function
to the choice of a "number"

- reliability first!
- maximize expected return (scaled?),
- max. E{return} & minimize customers lost,
- minimize Value-at-Risk (VaR),
- minimize the probability of any loss,
- minimizing a "Safeguarding" Measure
- variants & combinations of the above

Risk Measures

G. Pflug, A. Ruszczynski, W. Ogryczak, A. Shapiro

R.T. Rockafellar, S. Uryasev, M. Zabarankin

P. Artzner, F. Delbaen, J.-M. Eber and D. Heath

Set $\zeta = f(\xi, x)$ random returns, $\mathcal{Z} = \{\zeta\}$

Risk Function $\theta : \mathcal{Z} \to \overline{I\!R}$.

Concave Risk Measures

$\theta : \mathcal{Z} \to \overline{\mathbb{R}}$ *concave risk function* if

- Concavity: $\forall\, \boldsymbol{\zeta}, \boldsymbol{\eta} \in \mathcal{Z},\ \lambda \in [0, 1]$

$$\theta(\lambda\boldsymbol{\zeta} + (1-\lambda)\boldsymbol{\eta}) \geq \lambda\theta(\boldsymbol{\zeta}) + (1-\lambda)\theta(\boldsymbol{\eta})$$

- Monotonicity: $\boldsymbol{\zeta} \succ \boldsymbol{\eta} \implies \theta(\boldsymbol{\zeta}) \geq \theta(\boldsymbol{\eta})$
- Translation Equivariance:

$$\forall\, \alpha \in \mathbb{R}, \boldsymbol{\zeta} \in \mathcal{Z}, \theta(\alpha + \boldsymbol{\zeta}) = \alpha + \theta(\boldsymbol{\zeta})$$

Utility Theory

FUND. THM. With $F(\cdot;x)$ the returns' distribution of $f(\boldsymbol{\xi},x)$, suppose that for any feasible x^1, x^2: $F(\cdot;x^1)$ is either preferred or not (possibly indifferent) to $F(\cdot;x^2)$, then there exists a continuous, real-valued *utility function*

$$u : \mathbb{R} \to \mathbb{R} \quad \text{such that}$$

$$F(\cdot;x^1) \succ F(\cdot;x^2) \iff E\{u(f(\boldsymbol{\xi},x^1))\} > E\{u(f(\boldsymbol{\xi},x^2))\}$$

- knowing —or discovering— u
- u continuous, real-valued ("safeguarding" issues) consistent with continuous $\succ$

Practically: $\theta : \mathbb{R} \to \overline{\mathbb{R}}$ an *appraisal function*

Reliability: Chance Constraints

satisfy constraints with probability $\alpha \in (0, 1]$

$$\min f_0(x) \ \text{so that} \ \ \textsf{prob.}\ [\, x \in S(\xi)\,] \geq \alpha$$

Variant:

$$\begin{aligned} \min \quad & f_0(x) \\ \text{so that} \quad & \textsf{prob.}\ [\, f_i(\boldsymbol{\xi}, x) \leq 0\,] \geq \alpha_i, \ \ i \in I \end{aligned}$$

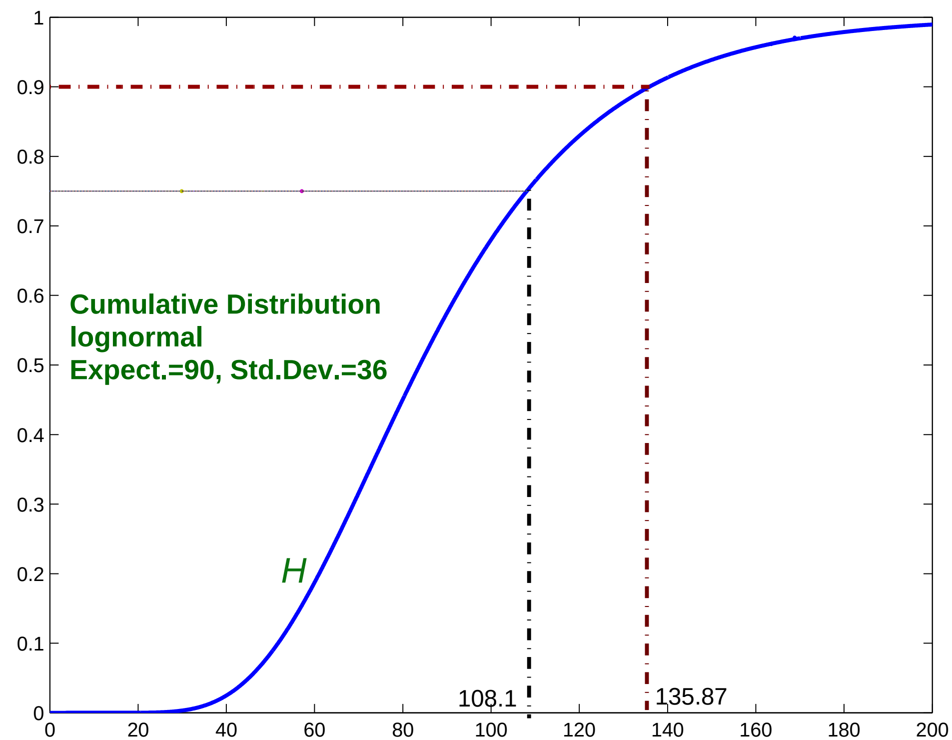
α_i dictated by ‘

- contractual obligations
- company policy, guess, etc.

Highly discontinuous appraisal function

Newsboy: Chance C. Model

$$\max -cx + (c+r)y \text{ so that } x \geq 0,\ y \in [H^{-1}(\alpha), x]$$



Infeasible if $x < H^{-1}(\alpha)$. Profit? not measured:

When $\alpha = 0.9 : \hat{x} = 135.9; \quad \alpha = 0.75 : \hat{x} = 108.1.$

VaR: Value-at-Risk

Let $F(s;x) = \text{prob}\,[\,-cx + Q(\boldsymbol{\xi}, x) \leq s\,]$

Value-at-Risk $\text{VaR}(\alpha; x)$: *"returns threshold"*.

$$\text{VaR}(\alpha; x) = F^{-1}(\alpha; x) \quad (= \sup\{v \,|\, v \in F^{-1}(\alpha; x)\})$$

$$\text{prob}[\text{ Returns } \leq \text{VaR}(\alpha; x)] \leq \alpha$$

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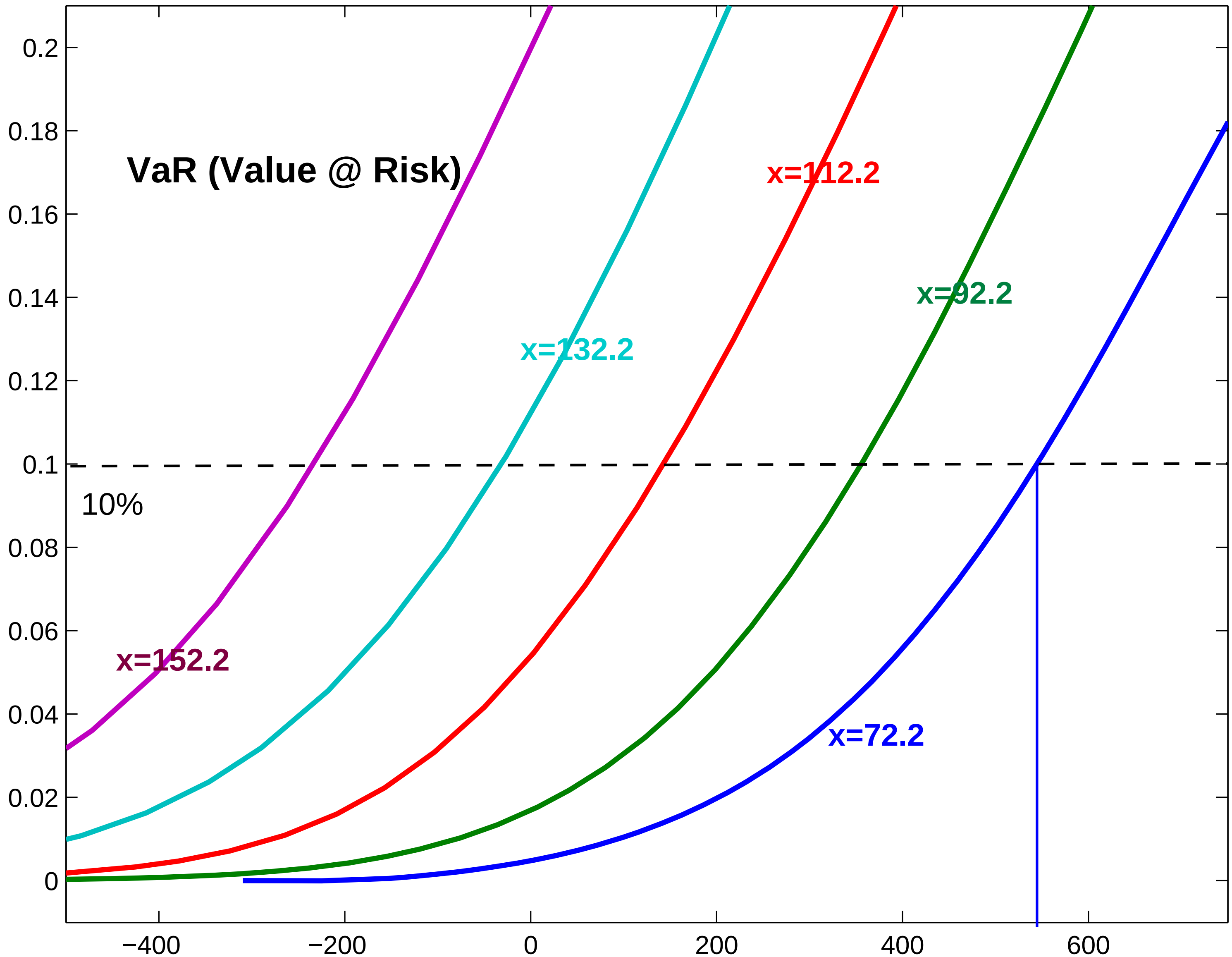
Objective: find x that maximizes $\mathrm{VaR}(\alpha;x)$

Newsboy: $\alpha = 0.10$

$$\mathrm{VaR}(.1;72.2) = 555, \cdots, \mathrm{VaR}(.1;152.2) = -226$$

VaR: Newsboy Problem



VaR: Properties

Challenge:

$$x \mapsto \mathrm{VaR}(\alpha; x)$$

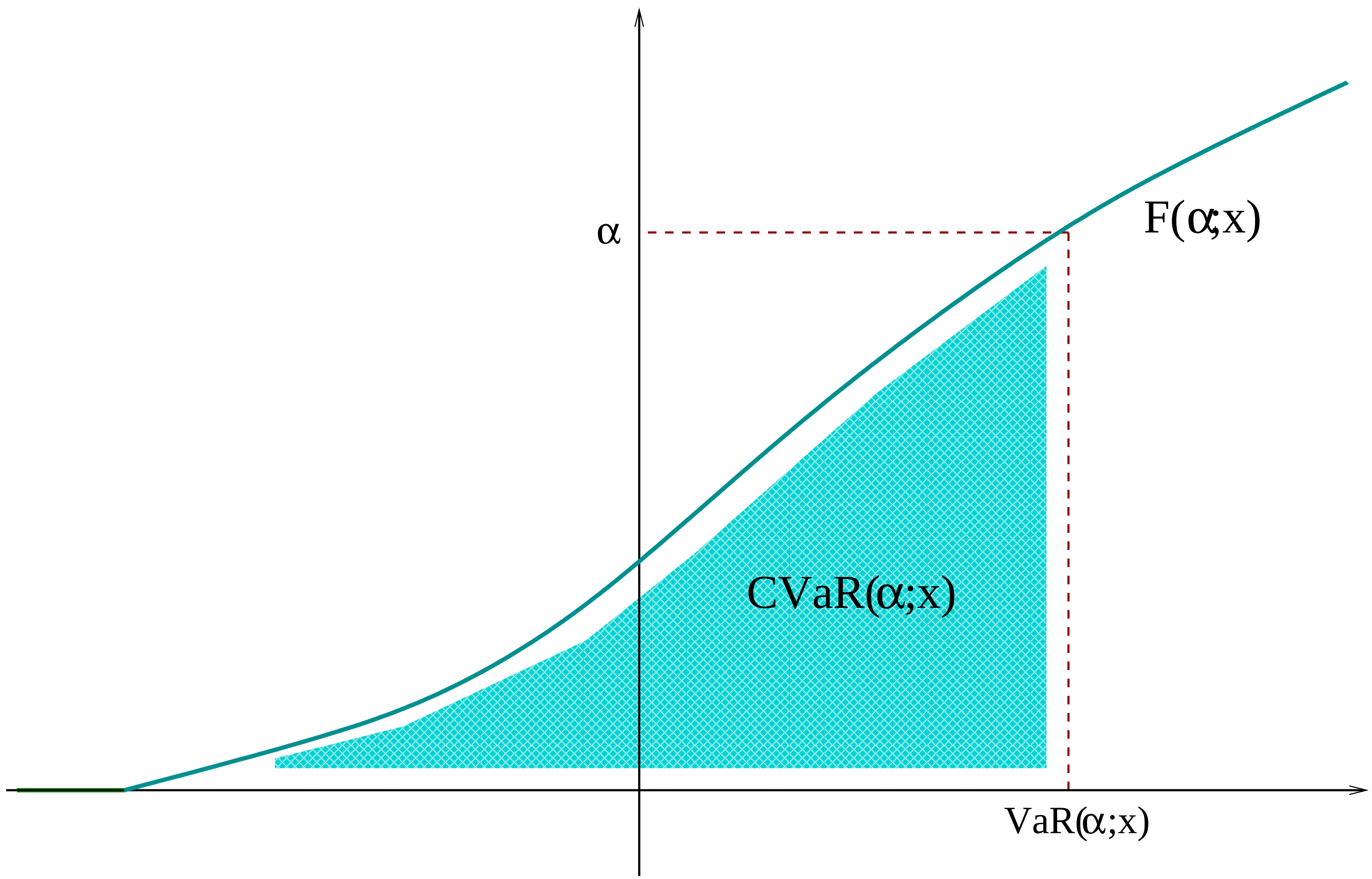
isn't concave (comp. unfriendly!) & unstable

Appraisal fcn:

$$\theta(f(\xi, x)) = \mathrm{VaR}(\alpha, x)$$

Unstable: $\alpha \mapsto \max \mathrm{VaR}(\alpha; \cdot)$ discontinuous and independent of ξ!

A Safeguarding Measure: CVaR



Conditional Value-at-Risk

Let $\zeta = f(\boldsymbol{\xi}, x) = -cx + Q(\boldsymbol{\xi}, x)$, $[t]^+ = \max\{0, t\}$

$$\mathrm{CVaR}(\alpha; x) = E\{\zeta \,\big|\, \zeta \leq \mathrm{VaR}(\alpha; x)\}$$

$$G(z; \alpha, x) = z + \frac{1}{1-\alpha} E\{[f(\boldsymbol{\xi}, x) - z]^+\}$$

THM. *For* $f(\xi, \cdot)$ *convex,* $\mathrm{CVaR}(\alpha; \cdot)$ *finite, concave, cont., i.e.,* argmax *comp. friendly!*

Proof. $\mathrm{CVaR}(\alpha; x) = \max_z -G(z; x, \alpha)$ and $(z, x) \mapsto G(z; \alpha, x)$ is jointly convex (definition) □

CVaR: Properties

$$x^* = \text{argmax}_x \, \text{CVaR}(\alpha; x)$$

$$\text{so that } (z^*, x^*) \in \text{argmax} -G(z; \alpha, x)$$

Stability. $\alpha \mapsto \text{CVaR}(\alpha; x)$ continuous with left- and right-hand derivatives [inf-projection argument]

Appraisal fcn: continuous, concave

$$\max_z \left([z + \frac{1}{1-\alpha}[-cx + Q(\boldsymbol{\xi}, x) - z]^+ \right)$$

$$\text{VaR}(\alpha; x) = \min \left\{ \text{argmin}_z -G(x; \alpha, x) \right\}$$

Penalties for "Shortcomings"

- ξ random demand for a (perishable) good
- $x \geq 0$ quantity ordered @ unit cost: $c = 10$
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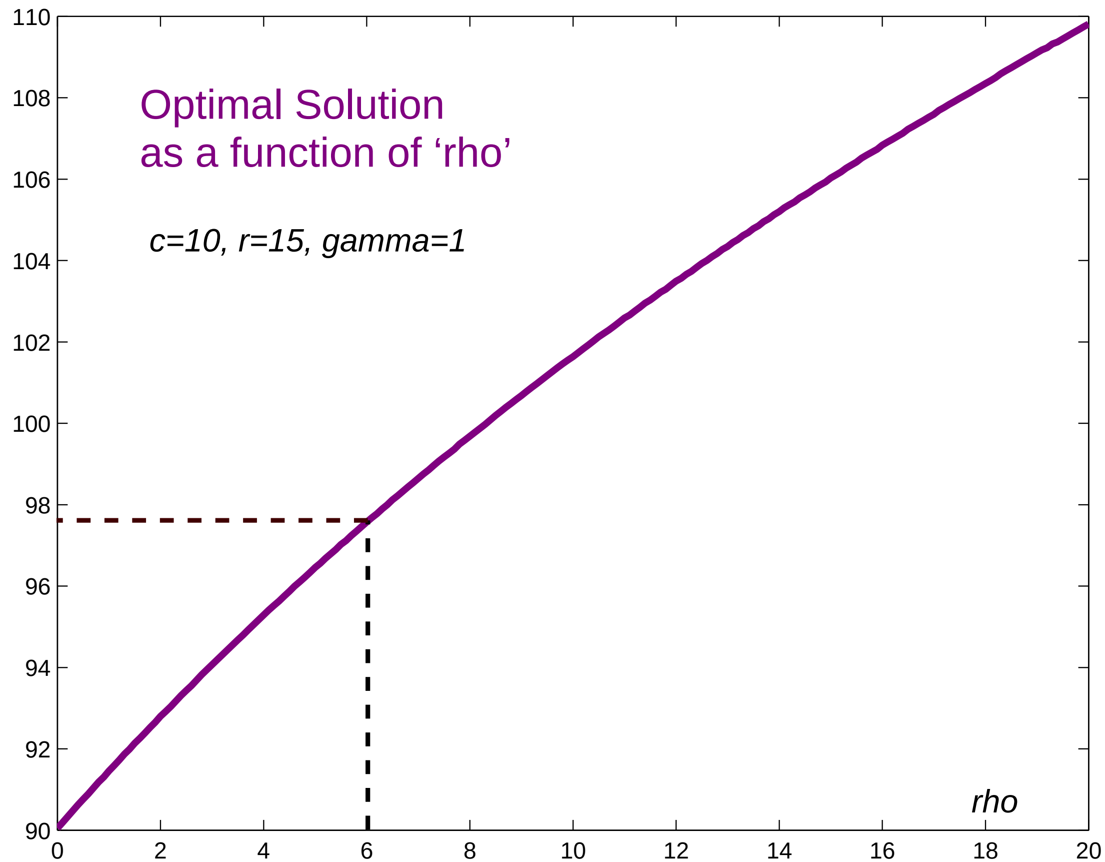
- ρ unit cost per lost-sale ('customer')
- $\gamma = 1$ unit disposal cost ('oversupply')

Expected Revenue: $-cx + E\{Q(\xi, x)\}$

$$Q(\xi, x) = (c + r)x - \gamma(x - \xi) \text{ if } \xi \leq x,$$
$$(c + r)\xi - \rho(\xi - x) \text{ if } \xi \geq x$$

Sol'n: Fcn of Penalty Parameters

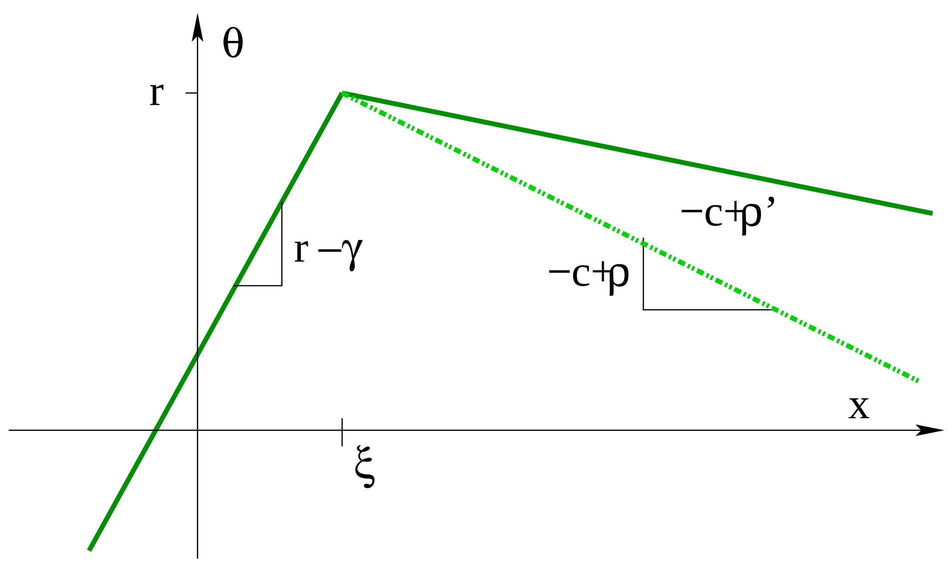
$$x^* = H^{-1}\Big(\frac{r+\rho}{c+r+\rho+\gamma}\Big) = H^{-1}\Big(\frac{15+\rho}{26+\rho}\Big)$$



with Penalty Param.: Properties

Appraisal fcn: continuous, concave

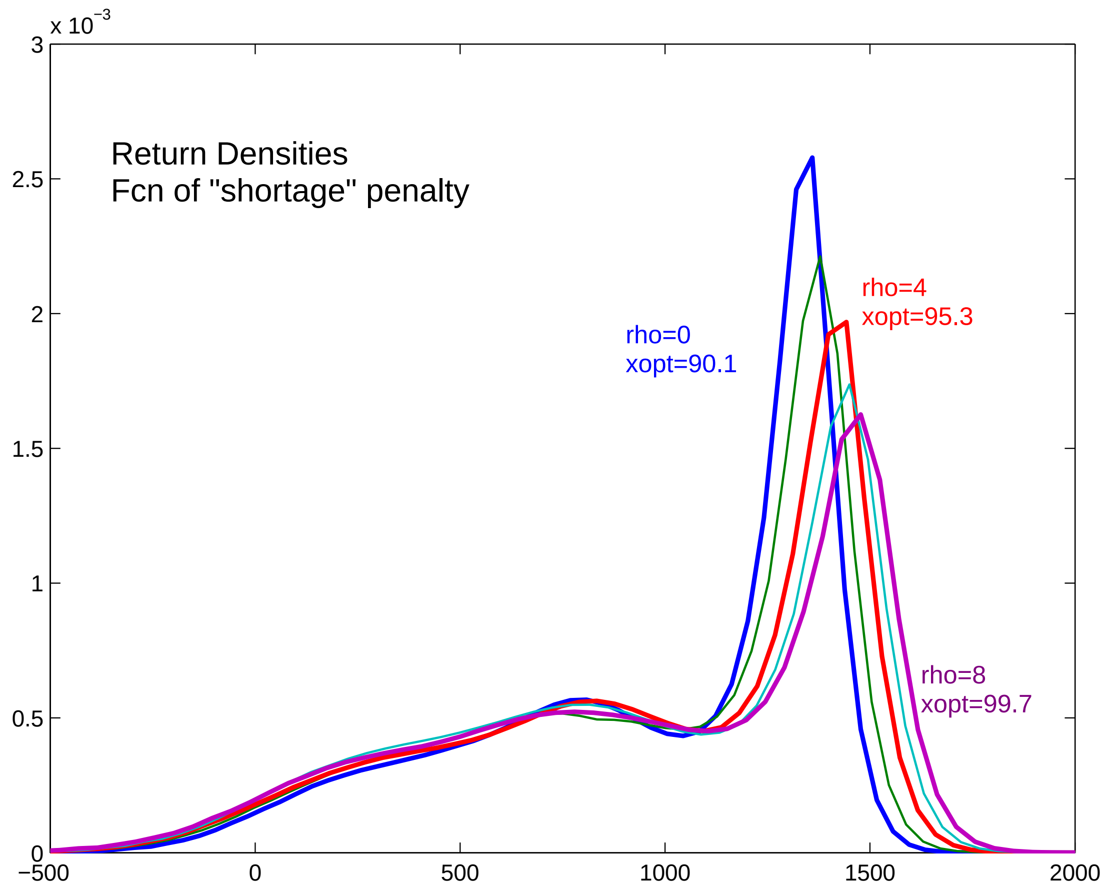
$$\theta(f(\xi, x)) = \begin{cases} rx + \gamma(\xi - x) & \text{if } \xi \leq x \\ rx + (c + r - \rho)(\xi - x) & \text{if } \xi \geq x \end{cases}$$



$\rho \mapsto x^*(\rho) = \text{argmax} - cx + E\{Q(\boldsymbol{\xi}, x; \rho)\}$ continuous.

with Penalty Param.: Newsboy

$$x^{opt}(\rho) = H^{-1}\left(\frac{r+\rho}{c+r+\gamma+\rho}\right) = H^{-1}\left(\frac{15+\rho}{26+\rho}\right)$$



Penalties & Chance Constraints

Newsboy: with C.C.: $x^{CC} = H^{-1}(\alpha)$

$$\max rx \;\text{ so that }\; x \geq \max\left[0, [H^{-1}(\alpha)\right]$$

with Penalty ρ: $x^P = H^{-1}\left(\frac{r+\rho}{r+c+\rho}\right)$

$$\max -cx + \int_0^x (c+r)\xi\, H(d\xi) + (c+r+\rho))x\, H(x)$$

Given $\alpha \; \exists$ corresponding ρ & vice-versa.

with $c = 10, r = 15$: $\alpha = .75 \implies \rho = 15$ ($\alpha = .5, \rho = 5$)

when $\rho = 10 \implies \alpha = 5/7 \approx .71$

Reliability & "Risk" Assessment

Start: $\min f_0(x)$ so that $Ax = b, Tx \geq d, x \geq 0$

demand $d = \boldsymbol{\xi}$ (& rand. T): distrib. fcn: F

Reliability: $\text{prob}\{Tx \geq \boldsymbol{\xi}\} \leq \alpha$.

$$\min f_0(x) \ : \ Ax = b, \ Tx = \chi, \ \chi \leq F^{-1}(\alpha), \ x \geq 0$$

Risk assessment: $\theta_i : I\!R \to I\!R_+$, $i = 1, \dots, m$

$$\min f_0(x) + \sum_{i=1}^{m} E\{\theta_i(\boldsymbol{\xi}_i - \chi_i)\} \ : \ Ax = b, Tx = \chi, x \geq 0$$

Reliability Model: Analysis

$$\min f_0(x) \;:\; Ax = b, Tx = \chi, \chi \leq F^{-1}(\alpha), x \geq 0$$

Optimality Conditions:

(a) $x^R \in \text{argmin}_{x\geq 0}[f_0(x) - \langle A^\top u + T^\top v, x\rangle]$

(b) $\chi^R \in \text{argmin}\langle v - \pi, \chi\rangle \quad \Longrightarrow \quad v = \pi$

(c) $\pi_i \leq 0$, $\chi_i^R - F_i^{-1}(\alpha)) \leq 0$

$$\pi_i(\chi_i^R - F_i^{-1}(\alpha)) = 0,\ i = 1, \ldots, m$$

Risk evaluation model

Appraisal functions: $\theta_i(\tau) = \begin{cases} 0 & \text{if } \tau \leq 0, \\ q_i \tau & \text{if } \tau > 0 \end{cases}$

Optimality Conditions:

(a) $x^* \in \operatorname{argmin}_{x \geq 0} [f_0(x) - \langle A^\top u + T^\top v, x \rangle]$

(b) $\chi_i^* \in \operatorname{argmin} v_i \chi_i + E\theta_i(\chi_i), \quad i = 1, \dots, m.$

$\implies \quad -v_i \in \partial E\theta_i(\chi_i^*) = q_i(1 - F_i(\chi^*))$

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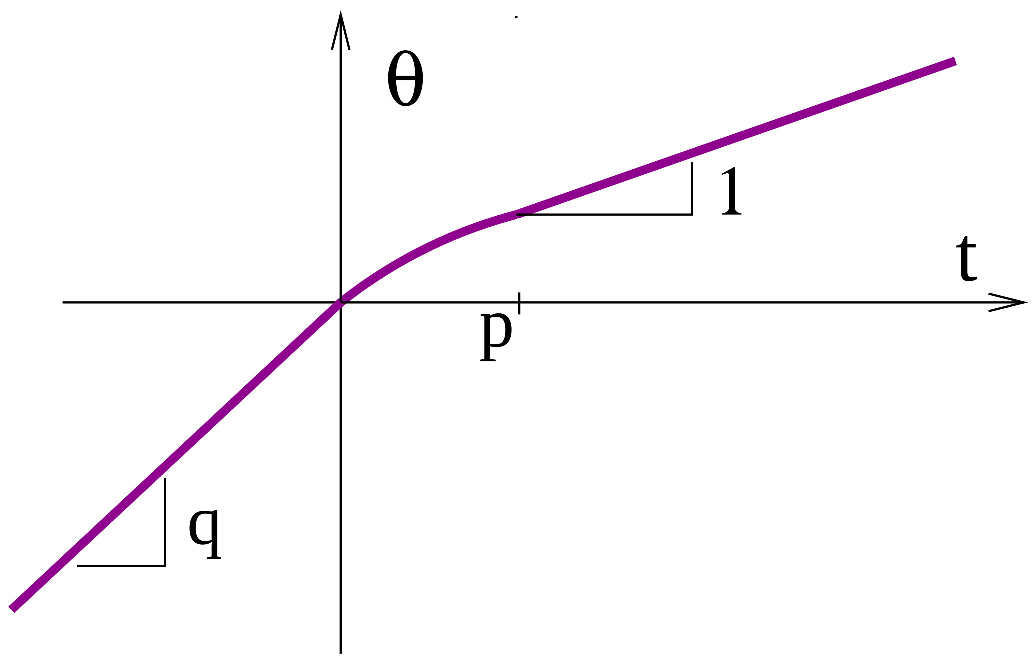
so, $x^* = x^R$ ($\chi^* = \chi^R$) if

$$q_i = -v_i^R / (1 - F_i(\chi^R)) > 0, \;\; i = 1, \dots, m$$

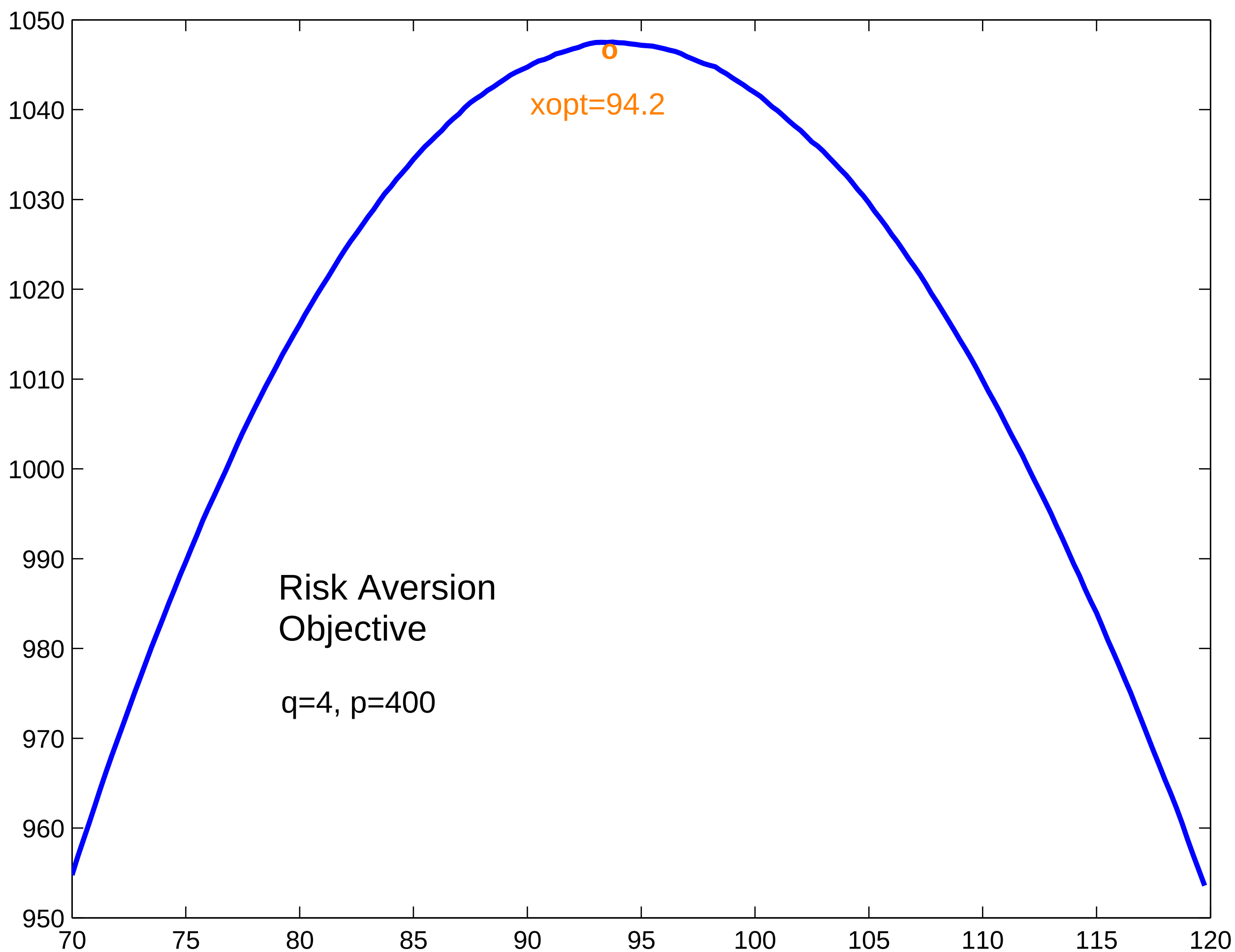
Conversely, given $q \quad \implies \quad \alpha^*$

Other Risk Aversion Functions

$$\theta(t) = \begin{cases} qt & \text{if } t \leq 0 \\ \frac{pq^2}{2(q-1)} - \frac{q-1}{2p}\left(t - \frac{qp}{q-1}\right)^2 & \text{if } t \in [0, p] \\ t & \text{if } t \geq p \end{cases}$$



Newsboy with Risk Aversion



Newsboy with Risk Aversion