

**RANDOM CLOSED SETS AND RANDOM LSC
FUNCTIONS:
KOLMOGOROV'S EXTENSION THEOREM ***

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Abstract. It is shown that Kolmogorov's Extension Theorem can be applied in the following nonstandard setting: A sequence $\{P^\nu, \nu \in \mathcal{N}\}$ of probability measures defined on $(\text{cl-sets}(X), \mathcal{E})$ that are consistent where X is a Polish space, $\text{cl-sets}(X)$ is the hyperspace of closed subsets of X and \mathcal{E} the Effrös field on $\text{cl-sets}(X)$. A similar result can be derived for a sequence of probability measures defined on an Effrös-like field built on the space of lsc functions defined on a Polish space.

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1. Introduction

Let (X, d) , be a Polish (complete, separable, metric) space and (Ξ, \mathcal{S}, P) a probability space. The *hyperspace of closed subsets* of X is denoted by $\text{cl-sets}(X)$ and $\text{lsc-fcns}(X)$ is the *space of lsc (lower semicontinuous) extended real-valued functions* defined on X . The purpose of this paper is to prove an appropriate version of the Kolmogorov's Extension Theorem for random variables, defined on (Ξ, \mathcal{S}) , with values in $\text{cl-sets}(X)$ or $\text{lsc-fcns}(X)$.

2. Probabilistic framework

We equip $\text{cl-sets}(X)$ with the *Effrös field* [4, 1], i.e., the σ -field \mathcal{E} generated by the following classes of subsets of $\text{cl-sets}(X)$:

$$\mathcal{F}_O = \{F \in \text{cl-sets}(X) \mid F \cap O \neq \emptyset\} \quad \text{with } O \text{ open .}$$

A *random closed set* S is a measurable function from Ξ into $\text{cl-sets}(X)$; *measurable* means that the inverse image of every Effrös measurable set is in \mathcal{S} , or equivalently, the inverse image of every set of the type \mathcal{F}_O is in \mathcal{S} with O an open subset of X . It will be convenient to identify S with the corresponding set-valued mapping from $\Xi \rightrightarrows X$ in which case measurability means that

$$\text{for any open set } O \subset X, \quad \{\xi \in \Xi \mid S(\xi) \cap O \neq \emptyset\} =: S^{-1}(O) \in \mathcal{S};$$

in particular, this implies that $\{\xi \in \Xi \mid S(\xi) \neq \emptyset\} = S^{-1}(X) =: \text{dom } S \in \mathcal{S}$. The *distribution* $P_S : \mathcal{E} \rightarrow [0, 1]$ of a random closed set S is defined by

$$P_S(\mathcal{A}) := P(\{\xi \in \Xi \mid S(\xi) \in \mathcal{A}\}) \quad \text{for } \mathcal{A} \in \mathcal{E}$$

and the *joint distribution* of a finite collection $\{S^1, \dots, S^k\}$ of random sets is then

$$P_{S^1, \dots, S^k}(\mathcal{A}_1, \dots, \mathcal{A}_k) := P(\{\xi \in \Xi \mid S^1(\xi) \in \mathcal{A}_1, \dots, S^k(\xi) \in \mathcal{A}_k\}).$$

for $\mathcal{A}_1, \dots, \mathcal{A}_k \in \mathcal{E}$.

For a sequence $\{S^\nu, \nu \in \mathbb{N}\}$ of random closed sets functions, let's denote by P^∞ the probability measure on the sequence space $(\text{cl-sets}(X)^\infty, \mathcal{E}^\infty)$ that is consistent with the joint distribution of the finite subcollections $\{S^{\nu_1}, \dots, S^{\nu_k}\}$. To assert the existence of such a measure, one usually appeals to Varadarajan's version of the Kolmogorov Extension Theorem, [8, Theorem V.5.1]:

Theorem 2.1. *Let $\{Y^\nu, \nu \in \mathbb{N}\}$ be random variables defined on (Ξ, \mathcal{S}, P) with values in a Polish space (Z, τ) and \mathcal{B} the associated Borel field on Z . Then there exists a unique*

measure μ^∞ on $(Z^\infty, \mathcal{B}^\infty)$ that is consistent with the joint distribution of $(Y^{\nu_1}, \dots, Y^{\nu_k})$ for any finite subcollection of indices $\{\nu_1, \dots, \nu_k\} \subset \mathbb{N}$.

However, this isn't quite our situation. At this stage we don't even have a topology on $\text{cl-sets}(X)$ that would allow us to contract the Borel field. Eventually, we are going to show that \mathcal{E} is the Borel field associated with certain topologies on $\text{cl-sets}(X)$ and with one of these, $\text{cl-sets}(X)$ is Polish.

3. Equivalence.

Both stationarity and ergodicity of sequences of random lsc functions have been stated in terms of the sequence space, $(\text{SC}(X)^\infty, \mathcal{F}^\infty, P^\infty)$. As mentioned, this all works out if there is a probability measure, P^∞ that is consistent with the joint distributions of the sequence. What's needed is to verify Kolmogorov's Extension Theorem. It will certainly hold on $(\text{SC}(X), \mathcal{F})$ if \mathcal{F} is the Borel field associated with a topology τ on $\text{SC}(X)$ such that $(\text{SC}(X), \tau)$ is a Polish space, cf. [3]. The remainder of this section is devoted to showing that the Borel σ -field associated with the topology of epi-convergence on $\text{SC}(X)$ satisfies this requirement.

For a Polish space Y , let $\text{cl-sets}(Y)$ denote the hyperspace of closed subsets of Y . A one-to-one correspondence exists between the space $\text{SC}(X)$ of lsc functions on X and the closed subsets of $X \times \mathbb{R}$ that are epigraphs. Under this correspondence, any σ -field \mathcal{F} of measurable subsets of $\text{SC}(X)$ can be considered as the restriction of a σ -field \mathcal{E} of measurable subsets of $\text{cl-sets}(X \times \mathbb{R})$ to the hyperspace of closed epigraphs. The results that follow are all therefore expressed in terms of $\text{cl-sets}(Y)$ for Y , a Polish space.

By definition, the *Effrös field* \mathcal{E} for the closed subsets of Y , is the σ -field on $\text{cl-sets}(Y)$ generated by all sets of the type

$$\mathcal{E}_D := \{C \in \text{cl-sets}(Y) \mid C \cap D \neq \emptyset\}, \quad D \subset Y, \quad D \text{ closed} .$$

When dealing with epigraphs, one can build the Effrös field by considering a restricted class of closed sets of the form $D \times (-\infty, \alpha]$ where D is now a closed subset of X . And when translating this to $\text{SC}(X)$, the generating sets take the form

$$A_{D,\alpha} = \{f \in \text{SC}(X) \mid \inf_D f \leq \alpha\}.$$

The Effrös field was introduced by E. Effrös [4]. It is the Borel field associated with various topologies of set convergence, including the *Fell topology*, which is a topology that corresponds to the epi-convergence of sequences of lsc functions, and the related *Choquet-Wijsman topology*. While the Fell topology is the right one for epi-convergence,

it is the Choquet-Wijsman topology that is guaranteed to be a Polish space, hence validating the Kolmogorov Extension Theorem. We elaborate on these statements in what follows.

Definition 3.3 (Fell topology). *The Fell topology τ_F on $\text{cl-sets}(Y)$ is the topology generated by a subbase consisting of sets of the form*

$$\mathcal{V}_O = \{C \in \text{cl-sets}(Y) \mid C \cap O \neq \emptyset\}, \quad O \subset Y, \quad O \text{ open}$$

and

$$\mathcal{V}_K = \{C \in \text{cl-sets}(Y) \mid C \cap K = \emptyset\}, \quad K \subset Y, \quad K \text{ compact} .$$

The Fell topology was introduced by Fell in 1962 [5]. That sequential τ_F -convergence of epigraphs corresponds to the epi-convergence of sequences of lsc functions on a first countable Hausdorff space (hence a Polish space) was first shown in [6], and a straightforward proof of this fact can be found in [1].

We shall write $\mathcal{B}(W, \tau)$ for the Borel field on the space W generated by the topology τ . First we establish that the Borel field for the Fell topology on $\text{cl-sets}(Y)$ contains the Effrös field.

Proposition 3.4. $\mathcal{E} \subset \mathcal{B}(\text{cl-sets}(Y), \tau_F)$.

Proof. This is an immediate consequence of the fact that the sets \mathcal{V}_O in the subbase for the Fell topology generate \mathcal{E} . □

Next we turn to the Choquet-Wijsman topology on $\text{cl-sets}(Y)$ which defined in terms of the *distance functions*: $d_C(x) := \inf_{y \in C} d(x, y)$, where d is the given metric on Y ; the convention applied for the distance to the empty set is $d_\emptyset(x) = +\infty$.

Definition 3.5 (cw-topology). *The Choquet-Wijsman topology on $\text{cl-sets}(Y)$ is the topology τ_{cw} generated by a subbase consisting of sets of the form*

$$\mathcal{W}_{x, \alpha, -} := \{C \in \text{cl-sets}(Y) \mid d_C(x) < \alpha\}, \quad x \in Y, \quad \alpha > 0,$$

and

$$\mathcal{W}_{x, \alpha, +} := \{C \in \text{cl-sets}(Y) \mid d_C(x) > \alpha\}, \quad x \in Y, \quad \alpha > 0.$$

R. Wijsman [9] and G. Choquet [2] introduced this notion of set convergence independently. In the Choquet-Wijsman topology, sequential convergence of sets is equivalent to the convergence of the distance functions: $C^\nu \xrightarrow{\tau_{cw}} C$ if and only if $d_{C^\nu}(x) \rightarrow d_C(x)$ for all $x \in Y$. For more about this topology, see [1]. It is known that the Choquet-Wijsman topology is finer than the Fell topology [1]. This immediately implies the inclusion, $\mathcal{B}(\text{cl-sets}(Y), \tau_F) \subset \mathcal{B}(\text{cl-sets}(Y), \tau_{cw})$.

Hess [7] showed that the Borel field generated by the Choquet-Wijsman topology is the Effrös field, but for the closed subsets of a separable space, Y , without the empty set. The empty set in $\text{SC}(X)$ corresponds to the function, $f \equiv \infty$; it therefore cannot be excluded from our considerations here. Nevertheless, it can be easily shown that $\mathcal{B}(\text{cl-sets}(Y), \tau_{cw}) = \mathcal{E}$ for all of $\text{cl-sets}(Y)$, by using a fact mentioned by Beer [1]. Let $\text{cl-sets}_{\neq \emptyset}(Y)$ denote the hyperspace of closed subsets of Y , not including the empty set.

Theorem 3.6. *Suppose τ is a topology on $\text{cl-sets}(Y)$ such that $\mathcal{B}(\text{cl-sets}_{\neq \emptyset}(Y), \tau) = \mathcal{E}$. Then $\mathcal{B}(\text{cl-sets}(Y), \tau) = \mathcal{E}$ if and only if $\{\emptyset\}$ is a Borel subset of $\text{cl-sets}(Y)$. In particular, this is true if $\{\emptyset\}$ is τ -closed.*

Theorem 3.7. $\mathcal{B}(\text{cl-sets}(Y), \tau_{cw}) = \mathcal{E}$.

Proof. We know from Hess [7] that $\mathcal{B}(\text{cl-sets}_{\neq \emptyset}(Y), \tau_{cw}) = \mathcal{E}$. So from the previous theorem, it suffices to verify that $\{\emptyset\}$ is τ_{cw} -closed, and this follows immediately from the fact that the complement of the empty set can be written as the union of open sets in $(\text{cl-sets}(Y), \tau_{cw})$: $\bigcup_{y \in Y} \bigcup_{\alpha \geq 0} \{C \in \text{cl-sets}(Y) \mid d_C(y) < \alpha\}$. And thus, $\{\emptyset\}$ is τ_{cw} -closed. \square

What has been shown is the following string of inclusions and equalities:

$$\mathcal{E} \subset \mathcal{B}(\text{cl-sets}(Y), \tau_F) \subset \mathcal{B}(\text{cl-sets}(Y), \tau_{cw}) = \mathcal{E}.$$

Therefore, the Borel fields for both the Fell topology and the Choquet-Wijsman topology coincide with the Effrös field as claimed.

It only remains to show that $(\text{cl-sets}(Y), \tau_{cw})$ is a Polish space. The proof is an extension, to include the empty set, of the following fact.

Theorem 3.8 [1]. $(\text{cl-sets}_{\neq \emptyset}(Y), \tau_{cw})$ is a Polish space.

Let $C(Y)$ denote the topological space of continuous functions from Y to \mathbb{R} under pointwise convergence. It will be necessary, given the metric, d , on Y , to consider the equivalent metric, $\tilde{d} := \arctan d$, so that in defining the point to set distance functions, the distance to the empty set is finite: $\tilde{d}_{\emptyset}(x) := \pi/2$. Different metrics on the space Y can lead to different Choquet-Wijsman topologies, but in this case these two are the same.

Proposition 3.9. $(\text{cl-sets}(Y), \tau_{cw-d}) = (\text{cl-sets}(Y), \tau_{cw-\tilde{d}})$.

Proof. Let A^λ be a net in $\text{cl-sets}(Y)$. Then, using the monotonicity of \arctan , we have that $cw-d\text{-}\lim_\lambda A^\lambda = A$

$$\begin{aligned}
&\iff \lim_{\lambda} d_{A^\lambda}(y) = d_A(y) && \text{(for all } y \in Y) \\
&\iff \lim_{\lambda} \inf_{x \in A^\lambda} d(x, y) = \inf_{x \in A} d(x, y) && \text{''} \\
&\iff \lim_{\lambda} \arctan \inf_{x \in A^\lambda} d(x, y) = \arctan \inf_{x \in A} d(x, y) && \text{''} \\
&\iff \lim_{\lambda} \inf_{x \in A^\lambda} \arctan d(x, y) = \inf_{x \in A} \arctan d(x, y) && \text{''} \\
&\iff \lim_{\lambda} \tilde{d}_{A^\lambda}(y) = \tilde{d}_A(y) && \text{''} \\
&\iff cw\text{-}\tilde{d}\text{-}\lim_{\lambda} A^\lambda = A. && \text{''}
\end{aligned}$$

□

Lemma 3.10 [1]. *Let (Y, d) and (X, d') be metric spaces, and let \mathcal{B} be an equicontinuous subfamily of $C(Y, X)$, the continuous functions from Y to X . Then*

(1) *The topology of uniform convergence on compact subsets of Y reduces to the topology of pointwise convergence on \mathcal{B} .*

(2) *The closure of \mathcal{B} in the topology of pointwise convergence (and thus in the finer topology of uniform convergence on compact sets) remains an equicontinuous family.*

(3) *If Y and X are second countable, then so is \mathcal{B} equipped with either function space topology.*

Theorem 3.11 (Arzelà-Ascoli). *Let \mathcal{B} be an equicontinuous family of functions from a separable space Y to a metric space X . Suppose that for each $y \in Y$, $\text{cl}\{f(y) \mid f \in \mathcal{B}\}$ is a compact subset of X . Then in \mathcal{B} , pointwise convergence is uniform on each compact subset of Y , and the closure of \mathcal{B} with respect to pointwise convergence (hence uniform convergence on compact sets) is compact.*

Lemma 3.12. *Let Y be a separable metric space. Then the closure, $\text{cl}(\text{cl-sets}(Y))$, of $\text{cl-sets}(Y)$ in $C(Y)$ under the identification $C \leftrightarrow \tilde{d}_C$, is a Polish space.*

Proof. The family of functions $\{\tilde{d}_C \mid C \in \text{cl-sets}(Y)\}$ is equi-lipschitzian and both Y and \mathbb{R} are separable metric spaces, so by Lemma 3.10, $\text{cl}(\text{cl-sets}(Y))$ is second countable and equicontinuous. We next use the fact that the functions in $\text{cl}(\text{cl-sets}(Y))$ are nonnegative and bounded above by $\pi/2$. This implies that $\text{cl}(\text{cl-sets}(Y))$ is pointwise bounded. Now, by the Arzela-Ascoli Theorem 3.11, $\text{cl}(\text{cl-sets}(Y))$ is compact in $C(Y)$. Thus, $\text{cl}(\text{cl-sets}(Y))$ is a second countable compact Hausdorff space. This implies it's completely metrizable, hence Polish. □

Theorem 3.13 (Alexandroff's Theorem [1]). *A Hausdorff space is completely metrizable if and only if it can be imbedded as a G_δ subset of some completely metrizable space.*

Lemma 3.14. *For sets $A_k \searrow$ and $B_k \searrow$,*

$$\bigcap_k [A_k \cup B_k] = \left[\bigcap_k A_k \right] \cup \left[\bigcap_k B_k \right].$$

Proof. \supset is trivial. For \subset , let $s \in \bigcap_k [A_k \cup B_k]$. Then $s \in A_k \cup B_k$ for all k . We will show that either $s \in A_k$ for all k or $s \in B_k$ for all k . Suppose not. Then by the monotonicity of A_k and B_k , there exist k, l such that $s \notin A_{\bar{k}}$ for all $\bar{k} \leq k$, and $s \notin B_{\bar{l}}$ for all $\bar{l} \leq l$. Let $m = \min \{k, l\}$. Then $s \notin A_m$ and $s \notin B_m$, whereby $s \notin A_m \cup B_m$, which contradicts the choice of s . \square

Theorem 3.15. $(\text{cl-sets}(Y), \tau_{cw})$ is a Polish space.

Proof. By Lemma 3.12 and Alexandroff's Theorem 3.13, it is enough to show that $\text{cl-sets}(Y)$ is a G_δ subset of its closure in $C(Y)$. Let $R = \{x_l\}$, $l \in \mathbb{N}$ be a countable dense subset of Y . For every $k, l \in \mathbb{N}$, let $\Gamma_{k,l}$ be the subset of $\text{cl}(\text{cl-sets}(Y))$ (in $C(Y)$) defined by

$$\Gamma_{k,l} := A_{k,l} \cup B_{k,l},$$

where

$$A_{k,l} := \{f \in \text{cl}(\text{cl-sets}(Y)) \mid f^{-1}[0, 2^{-k-1}) \neq \emptyset \text{ and } \tilde{d}_{f^{-1}[0, 2^{-k-1})}(x_l) < f(x_l) + 2^{-k}\},$$

$$B_{k,l} := \{f \in \text{cl-sets}(Y) \mid \tilde{d}_{f^{-1}(\pi/2 - 2^{-k-1}, \pi/2]}(x_l) < 2^{-k}\}.$$

It can be easily shown that each such set, $\Gamma_{k,l}$, is open in $\text{cl}(\text{cl-sets}(Y))$. We next claim that

$$\text{cl-sets}(Y) = \bigcap_{l \in \mathbb{N}} \bigcap_{k \in \mathbb{N}} \Gamma_{k,l}.$$

Beer, in his proof of Theorem 3.8, shows that

$$\text{cl-sets}_{\neq \emptyset}(Y) = \bigcap_{l \in \mathbb{N}} \bigcap_{k \in \mathbb{N}} A_{k,l}.$$

It suffices, therefore, to show that

$$\{f \equiv \pi/2\} = \bigcap_{l \in \mathbb{N}} \bigcap_{k \in \mathbb{N}} B_{k,l}$$

and that

$$\bigcap_{l \in \mathbb{N}} \bigcap_{k \in \mathbb{N}} [A_{k,l} \cup B_{k,l}] = \left[\bigcap_{l \in \mathbb{N}} \bigcap_{k \in \mathbb{N}} A_{k,l} \right] \cup \left[\bigcap_{l \in \mathbb{N}} \bigcap_{k \in \mathbb{N}} B_{k,l} \right].$$

To show that $\bigcap_l \bigcap_k B_{k,l} = \{f \equiv \pi/2\}$, observe that \supset is clear, since $f \equiv \pi/2$ is in $\text{cl-sets}(Y)$, and $\tilde{d}_{f^{-1}(\pi/2 - 2^{-k-1}, \pi/2]}(x_l) = \tilde{d}_Y(x_l) = 0 < 2^{-k}$. For the \subset direction, let $f \in \bigcap_l \bigcap_k B_{k,l}$. Then for all k and for all l ,

$$\tilde{d}_{f^{-1}(\pi/2 - 2^{-k-1}, \pi/2]}(x_l) < 2^{-k},$$

Suppose that $f \neq \pi/2$. Then by the continuity of f , there is an x_l such that $f(x_l) = \gamma$ and $|\gamma - \pi/2| = \varepsilon > 0$. Again by the continuity of f , there is a $\delta > 0$ such that for all $\bar{x} \in \mathbb{B}(x_l, \delta)$, $|f(\bar{x}) - \gamma| < \varepsilon/2$. Now, choose k such that $2^{-k-1} < \varepsilon/2$ and $2^{-k} < \delta$. Then

$$f^{-1}(\pi/2 - 2^{-k-1}, \pi/2] \subset f^{-1}(\pi/2 - \varepsilon/2, \pi/2].$$

So,

$$\tilde{d}_{f^{-1}(\pi/2 - 2^{-k-1}, \pi/2]}(x_l) \geq \tilde{d}_{f^{-1}(\pi/2 - \varepsilon/2, \pi/2]}(x_l) \geq \delta > 2^{-k}.$$

This contradicts $f \in \bigcap_l \bigcap_k B_{k,l}$.

Next we turn to showing that the countable intersection of the unions is the union of the countable intersections. We show first that for fixed l ,

$$\bigcap_k [A_{k,l} \cup B_{k,l}] = \left[\bigcap_k A_{k,l} \right] \cup \left[\bigcap_k B_{k,l} \right].$$

This is true by virtue of the fact that as $k \rightarrow \infty$, $A_{k,l} \searrow$ and $B_{k,l} \searrow$, and Lemma 3.14. We finish by showing that

$$\bigcap_l \left(\left[\bigcap_k A_{k,l} \right] \cup \left[\bigcap_k B_{k,l} \right] \right) = \left[\bigcap_l \bigcap_k A_{k,l} \right] \cup \left[\bigcap_l \bigcap_k B_{k,l} \right].$$

Fix l , and suppose that $f \in \left[\bigcap_k A_{k,l} \right] \cup \left[\bigcap_k B_{k,l} \right]$. We use the easily verified fact that $\bigcap_k B_{k,l} = \{f \in \text{cl}(\text{cl-sets}(Y)) \mid f(x_l) = \pi/2\}$. Suppose $f \in \bigcap_k B_{k,l}$. Then we claim that either (1) $f \in \bigcap_k A_{k,l}$ or (2) $f \notin \bigcap_k A_{k,l}$ for all l . (1) is true if for all k , $f^{-1}[0, 2^{-k-1}) \neq \emptyset$, since then

$$\tilde{d}_{f^{-1}[0, 2^{-k-1})}(x_l) \leq \pi/2 = f(x_l) \leq f(x_l) + 2^{-k-1}.$$

(2) is true in the remaining cases, when there exists a k such that $f^{-1}[0, 2^{-k-1}) = \emptyset$, since then by definition of $A_{k,l}$, $f \notin \bigcap_k A_{k,l}$ for any l . Now, suppose $f \in \bigcap_l \left(\left[\bigcap_k A_{k,l} \right] \cup \left[\bigcap_k B_{k,l} \right] \right)$. Then for all l , $f \in \left[\bigcap_k A_{k,l} \right] \cup \left[\bigcap_k B_{k,l} \right]$. What we have just shown in (1) and (2) implies that either $f \in \bigcap_k A_{k,l}$ for all l , or $f \in \bigcap_k B_{k,l}$ for all l , whereby

$$f \in \left[\bigcap_l \bigcap_k A_{k,l} \right] \cup \left[\bigcap_l \bigcap_k B_{k,l} \right].$$

This establishes that $(\text{cl-sets}(Y), \tau_{cw})$ is a G_δ subset of a Polish space, and hence by Alexandroff's Theorem 3.13, that it is itself Polish. \square

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