EPIGRAPHICAL PROCESSES: LAWS OF LARGE NUMBERS FOR RANDOM LSC FUNCTIONS †

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Abstract. New laws of large numbers are obtained for random lower semicontinuous functions. The bicontinuity of the Legendre-Fenchel transform is used to exhibit the connection to earlier results for random sets of Artstein-Vitale and Hess-Hiai.

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A new law of large numbers for random lower semicontinuous functions is formulated and proved when convergence to the limit function is in terms of epi-convergence, the basic convergence notion for variational problems; epi-convergence induces, in a sense that can be made precise, the convergence of infima and minimizers. When the random lower semicontinous functions are also *convex*, laws of large number for more restrictive notions of convergence are also derived, and in one of these cases (Mosco-epi-convergence) we make the connection with the laws of large numbers for random sets (Artstein and Vitale [**3**], Artstein and Hart [**2**], Hiai [**25**] and Hess [**23**]). Part of the motivation for this work comes on one end from a variety of potential applications to problems involving random composites, porous media, and more generally, nonhomogeneous materials that are "stochastically" mixed [**18**], and on the other end to supply justification for the use of sampling in the design of solution procedures for stochastic optimization problems [**26**].

1. Preliminaries

We work with the following set up: let $(X, |\cdot|)$ be a separable Banach space, $\mathcal{B}$ the Borel field generated by the open subsets of X, and μ a probability measure on the measure space $(\Omega, \mathcal{A})$ with $\mathcal{A}$ μ-complete. A *set-valued mapping* Γ with domain Ω and with values in the subsets of X,

$$\omega \mapsto \Gamma(\omega) : \Omega \rightrightarrows X$$

is a *random closed set*, if for all ω, $\Gamma(\omega)$ is closed and for any open set $G \subset X$,

$$\Gamma^{-1}(G) := \{\omega \in \Omega \,\big|\, \Gamma(\omega) \cap G \neq \emptyset\} \in \mathcal{A}.$$

Let $\mathcal{F} = \mathcal{F}(X)$ denote the class of closed subsets of X with $\mathcal{F}_0(X)$, the class of closed nonempty subsets of X; note that the empty set is a member of $\mathcal{F}$. The *Effros* field on $\mathcal{F}$ is the σ-field $\mathcal{S}$ generated by all sets of the type

$$\mathcal{F}_G := \{F \in \mathcal{F} \,\big|\, F \cap G \neq \emptyset\}, \quad G \subset X \text{ open.}$$

By identifying every closed subset of X with the corresponding element in $\mathcal{F}$, we can associate with every random closed set Γ a random variable, i.e., a $\mathcal{S}$-measurable mapping, $\gamma : \Omega \to \mathcal{F}$. We denote by $\mathcal{A}_\Gamma$ the σ-field induced by Γ (or equivalently γ) on the measure space $(\Omega, \mathcal{A})$; we have $\mathcal{A}_\Gamma = \gamma^{-1}(\mathcal{S})$.

Two random closed sets Γ_1, Γ_2 are *independent* if the induced σ-fields are independent, i.e., if for all $A_1 \in \mathcal{A}_{\Gamma_1}$ and $A_2 \in \mathcal{A}_{\Gamma_2}$:

$$\mu(A_1 \cap A_2) = \mu(A_1) \cdot \mu(A_2).$$

A collection of random closed sets $\{\Gamma_\nu, \nu\ I\!\!N\}$ are said to be *independent* if the random closed sets of any finite subcollection are independent, and they are *pairwise independent* if any pair of random closed sets $\Gamma_{\nu_1}, \Gamma_{\nu_2}$ are independent.

The *distribution* of a random set Γ is the probability measure induced by the random variable γ on $(\mathcal{F}, \mathcal{S})$; we denote those distributions by P, P_Γ, etc.. Let's observe that $\gamma^{-1}(\mathcal{F}_G) = \Gamma^{-1}(G)$ for any open set $G \subset X$, and thus $P_\Gamma(\mathcal{F}_G) = \mu[\Gamma^{-1}(G)]$. Convergence in distribution of random closed sets means narrow (weak) convergence of the induced distributions on $(\mathcal{F}, \mathcal{S})$; for more about convergence of random sets, refer to [**30**]. The random closed sets of a collection are said to be *identically distributed* if they induce the same distribution on $(\mathcal{F}, \mathcal{S})$. We say that the collection consists of *iid* random sets if they are both independent and identically distributed, and they are said to be *piid* if they are pairwise independent and identically distributed.

An extended real-valued function f defined on $X \times \Omega$ is a *random lower semicontinuous (lsc) function* if its epigraphical set-valued mapping,

$$\omega \mapsto \operatorname{epi} f(\cdot, \omega) : \Omega \rightrightarrows X \times \mathbb{R}$$

is a random closed set; recall that the *epigraph* of a function $g : X \to \overline{\mathbb{R}}$ is the set $\operatorname{epi} g = \{(x, \alpha) \in X \times \mathbb{R} \,\big|\, g(x) \leq \alpha\}$. The concept of a random lsc function is due to Rockafellar [**29**] who introduced it in the context of the calculus of variations under the name of "normal integrand."

Taking into account the fact that X is separable, and that $(x, \beta) \in \operatorname{epi} f(\cdot, \omega)$ implies $(x, \beta') \in \operatorname{epi} f(\cdot, \omega)$ for all $\beta' \geq \beta$, the measurability of $\omega \mapsto \operatorname{epi} f(\cdot, \omega)$ can also expressed in the following terms:

$$\forall \text{ open } G \subset X, \alpha \in \mathbb{R}, \qquad \{\omega \,|\, \inf_G f(\cdot, \omega) < \alpha\} \in \mathcal{A}.$$

Thus $f : X \times \Omega \to \overline{\mathbb{R}}$ is a random lsc function if

(i) for all $\omega \in \Omega$, the function $x \mapsto f(x, \omega)$ is lower semicontinuous,
(ii) for all open $G \subset X$, the function $\omega \mapsto \inf_G f(\cdot, \omega) : \Omega \to \overline{\mathbb{R}}$ is $\mathcal{A}$-measurable.

From the definition, it also follows that for a (bivariate) function $f : X \times \Omega \to \overline{\mathbb{R}}$ to be a random lsc function it is necessary that

(i) for all $\omega \in \Omega$, the function $x \mapsto f(x, \omega)$ is lower semicontinuous.
(ii) $(x, \omega) \mapsto f(x, \omega)$ is $(\mathcal{B} \otimes \mathcal{A})$-measurable.

These conditions are also sufficient if $\mathcal{A}$ is μ-complete, cf. [**16**, lemma VII.1, theorem III.30]. In particular, this implies immediately that in the present setting (X a separable Banach space, $\mathcal{A}$ μ-complete) the sum of two, or more generally, of any finite or countable number of random lsc functions, is again a random lsc function.

The space $\operatorname{SC}(X)$ of lower semicontinuous functions defined on X is in one-to-one correspondence with the space of closed subsets of $X \times \mathbb{R}$ that are epigraphs; a set $E \subset X \times \mathbb{R}$ is an epigraph if $(x, \beta) \in E \Rightarrow E \cap (\{x\} \times \mathbb{R})$ is either $\mathbb{R}$ or $[\tilde{\beta}, \infty)$ for some $\tilde{\beta} \leq \beta$; $\operatorname{SC}_0(X)$ denotes the space of *proper lsc functions*, i.e., $f \in \operatorname{SC}_0(X)$ implies f is lsc,

$f > -\infty$ and $f \not\equiv \infty$. The Effros field on $\mathrm{SC}(X)$ is the σ-field generated by all sets of the type

$$\{\{f \in \mathrm{SC}(X) \mid \inf_G f < \alpha\},\ G \text{ open },\alpha \in \mathbb{R}\}.$$

In §3, we make the connection between the Effros field and the Borel structures associated with certain topologies that can be defined on $\mathrm{SC}(X)$.

The distribution P_f of a random lsc function f is the distribution associated with its epigraphical mapping, i.e., $P_f = P_{\mathrm{epi}\, f}$. We denote by $\mathcal{A}_f$ for the σ-field induced by the random lsc function f, we have $A_f = A_{\mathrm{epi}\, f}$. Narrow convergence of random lsc functions is defined in terms of the narrow convergence of their distributions. For more about convergence of random lsc functions, consult [**30**]. Random lower semicontinuous functions are *independent* if the induced σ-fields are independent, or equivalently, if the associated epigraphical mappings are independent random closed sets; a collection is said to be *pairwise independent* if any two are independent. They are *identically distributed* if the distributions induced by their epigraphical mappings are identical. And, one says that a collection of random lsc functions is *iid* if its members possess both properties. Random lsc functions are *piid* if they are pairwise independent and identically distributed.

The correspondence between functions and sets, via the epigraphical mapping $f \mapsto \mathrm{epi}\, f$, leads to a number of operations on functions whose motivations come from the effect they have on epigraphs. The *epi-sum* of two functions $g_1, g_2 : X \to \overline{\mathbb{R}}$ is defined by:

$$(g_1 \mathbin{+_{\mathrm{e}}} g_2)(x) := \inf_{u \in X} [g_1(u) + g_2(x-u)];$$

this operation is also called *inf-convolution*. The use of the prefix "epi" and the subscript "e" refers to the fact that this operation (addition, in this case) is taking place on epigraphs. Indeed, $\mathrm{epi}(g_1 \mathbin{+_{\mathrm{e}}} g_2)$ is the "vertical-closure" of the sum of the epigraphs of g_1 and g_2:

$$\mathrm{epi}(g_1 \mathbin{+_{\mathrm{e}}} g_2) = \{(x, \inf \alpha) \mid (x, \alpha) \in \mathrm{epi}\, g_1 + \mathrm{epi}\, g_2\}.$$

For $\alpha > 0$, the *epi-multiple* of a function $g : X \to \overline{\mathbb{R}}$ is the function $(\alpha *_{\mathrm{e}} g) : X \to \overline{\mathbb{R}}$, with the operation "$*_{\mathrm{e}}$" defined by $\alpha *_{\mathrm{e}} g(x) = \alpha g(\alpha^{-1} x)$. It follows that $\alpha\, \mathrm{epi}\, g = \mathrm{epi}(\alpha *_{\mathrm{e}} g)$. Epi-addition has generally a smoothing effect. We are going to consider the epi-sum of arbitrary functions and a "kernel" of the type $(1/p)|\cdot|^p$.

Given $p \in [1, \infty)$ and $\lambda > 0$, the *epi-regularization* (of order p) of index λ of $g : X \to \overline{\mathbb{R}}$ is the function

$$g_\lambda := g \mathbin{+_{\mathrm{e}}} \frac{1}{\lambda p} |\cdot|^p;$$

i.e., for all $x \in X$:

$$g_\lambda(x) = \inf_{u \in X} \left[g(u) + \frac{1}{\lambda p} |x-u|^p \right].$$

When $p = 1$, g_λ is called the Pasch-Hausdorff epi-regularization, and when $p = 2$, it is called the Moreau-Yosida epi-regularization. The properties of the g_λ are summarized in the following lemma.

Lemma 1.1. [**10**, lemma 3.2], [**9**, section 3]. *Let $g : X \to \overline{\mathbb{R}}$ be an extended real-valued function such that $g \not\equiv \infty$ and for some $x_0 \in X$, $\alpha_0 \geq 0$ and $\alpha_1 \in \mathbb{R}$,*

$$g \geq -\alpha_0 |\cdot - x_0|^p - \alpha_1$$

for some $p \in [1, \infty)$. Then, the family of the epi-regularizations of g,

$$\{g_\lambda = g \mathbin{+_{\mathrm{e}}} (\lambda p)^{-1} |\cdot|^p, \lambda > 0\},$$

is endowed with the following properties:

(a) *for all $x \in X$ and all $\lambda > 0$, the sequence $\{g_\lambda(x)\}_{\lambda>0}$ increases to $\operatorname{cl} g(x)$ (lsc-regularization of g) as λ decreases to 0; thus, if g is lsc, $g = \sup_{\lambda>0} g_\lambda$;*

(b) *for all λ sufficiently close to 0 ($0 < \lambda < (\alpha_0 p)^{-1} 2^{1-p}$), $x \mapsto g_\lambda(x)$ is a real-valued lsc function; in fact, g_λ is Lipschitz continuous on any bounded set, i.e.*

$$|g_\lambda(x) - g_\lambda(x')| \leq \lambda^{-1} \kappa |x - x'|$$

where the constant κ depends continuously on $|x|$, $|x - x'|$, α_0, λ and p, it depends on g only through the value $g(x_g)$ and $|x_g|$ of a point x_g at which g is finite.

The lower bound for g_λ can be obtained as follows:

$$\begin{aligned} g_\lambda(x) &= \inf_{u \in X} \{ g(u) + \frac{1}{\lambda p} |x - u|^p \} \\ &\geq \inf_{u \in X} \{ -\alpha_0 |u - x_0|^p - \alpha_1 + \frac{1}{\lambda p} |x - u|^p \}. \end{aligned}$$

Since

$$|u - x_0|^p \leq 2^{p-1} \big(|u - x|^p + |x - x_0|^p \big)$$

one has

$$\begin{aligned} g_\lambda(x) &\geq \inf_{u \in X} \{ -\alpha_0 2^{p-1} |u - x|^p - \alpha_0 2^{p-1} |x - x_0|^p + \frac{1}{\lambda p} |x - u|^p \} - \alpha_1 \\ &\geq \inf_{u \in X} \{ (\frac{1}{\lambda p} - \alpha_0 2^{p-1}) |u - x|^p \} - 2^{p-1} \alpha_0 |x - x_0|^p - \alpha_1. \end{aligned}$$

Now, $(1/\lambda p) > \alpha_0 2^{p-1}$ ($0 < \lambda < 2^{1-p} (\alpha_0 p)^{-1}$), it follows

$$g_\lambda(x) \geq -2^{p-1} \alpha_0 |x - x_0|^p - \alpha_1.$$

When $p = 1$, g_λ is Lipschitz continuous on the whole of X, but there is then no guarantee that g_λ will be differentiable. On the other hand, when $p = 2$, g_λ is only locally Lipschitz continuous, but it then enjoys a number of desirable regularity properties. For example, when $p = 2$, X is a Hilbert space, and g is lsc convex, then g_λ is smooth (continuously differentiable), a property that will be exploited in sections 5 and 6. Epigraphical regularization supplies a powerful tool for the analysis of random lsc functions. This was already exploited by Dal Maso and Modica [**18**] in their work on stochastic homogenization involving random integral functionals.

We show next that if random lsc functions are iid, so are their epi-regularizations (with respect to the x-variable), and the (vector-valued) random variables generated by their restrictions to a finite subset of X are iid.

Lemma 1.2. *Suppose* Γ_1 *and* Γ_2 *are iid random closed sets with values in the closed subsets* $\mathcal{F}$ *of a normed linear space* X *and let* D *be an arbitrary closed subset of* X. *Then, the closed random sets* $\mathrm{cl}(\Gamma_1 + D)$ *and* $\mathrm{cl}(\Gamma_2 + D)$ *are also iid random closed sets.*

Proof. Let $\Gamma_i' = \mathrm{cl}(\Gamma_i + D)$ for $i = 1, 2$. Let G be an open subset of X. Then

$$\begin{aligned}(\Gamma_i')^{-1}(G) &= \{\omega \,|\, (\Gamma_i(\omega) + D) \cap G \neq \emptyset\}\\ &= \{\omega \,|\, \Gamma_i(\omega) \cap (G + (-D)) \neq \emptyset\}\\ &= \Gamma_i^{-1}(G - D).\end{aligned}$$

Since $G - D$ is open, $(\Gamma_i')^{-1}(G) \in \mathcal{A}$, and thus Γ_i' is a random closed set. Moreover, it follows from these identities that the σ-field generated by Γ_i' are contained in those generated by Γ_i and since by assumption the σ-fields generated by Γ_1 and Γ_2 are independent, so are those generated by Γ_1' and Γ_2'.

Let P_i' be the distribution associated with the random closed set Γ_i'. In view of the above, we have that for open G:

$$P_i'(\mathcal{F}_G) = \mu[(\Gamma_i')^{-1}(G)] = \mu[\Gamma_i^{-1}(G - D)] = P_i(\mathcal{F}_{G-D}).$$

In fact, for any finite collection of open sets $G^k, k = 1, \ldots, q$, one verifies similarly that

$$P_i'\left(\bigcap_{k=1}^{q} \mathcal{F}_{G^k}\right) = P_i\left(\bigcap_{k=1}^{q} \mathcal{F}_{G^k - D}\right).$$

Because $\{\mathcal{F}_{G^1} \cap \cdots \cap \mathcal{F}_{G^q},\quad G^k \text{ open}, \ q \text{ finite}\}$ is a generating class for $\mathcal{S}$, closed under taking finite intersections, the preceding identity implies that the random closed sets Γ_1' and Γ_2' have the same distribution whenever Γ_1 and Γ_2 have the same distribution. □

Corollary 1.3. *Let* $\{f^\iota : X \times \Omega \to \overline{\mathbb{R}}, \iota \in I\}$ *be a collection of piid random lsc functions such that almost surely* $f^\iota \not\equiv \infty$ *and there exist* $p \in [1, \infty)$, x_0, $\alpha_0 \geq 0$ *and* $\alpha_1 \in \mathbb{R}$ *such that for all* $\iota \in I$,

$$f^\iota(\cdot, \omega) \geq -\alpha_0 |\cdot - x_0|^p - \alpha_1, \text{ for almost all } \omega.$$

Then, for all $\lambda > 0$ *sufficiently close to* 0 ($\lambda < (\alpha_0 p)^{-1} 2^{1-p}$), *the epi-regularizations (of order* p) f_λ^ι *are piid (real-valued) random lsc functions, minorized almost surely by* $-2^{p-1}\alpha_0 |\cdot - x_0|^p - \alpha_1$.

Proof. Fix $\lambda > 0$. For every $\iota \in I$, f_λ^ι is a random lsc function: the lower semicontinuity (and more) follows from lemma 1.1 using the fact that the random lsc functions are almost surely minorized $-\alpha_0|\cdot - x_0|^p - \alpha_1$, the measurability of the epigraphical set-valued mapping $\omega \mapsto \mathrm{epi}\, f_\lambda^\iota(\cdot, \omega)$ follows from the argument used in the proof of lemma 1.2 that shows that for all open subsets $G' \subset X \times \mathbb{R}$:

$$(\mathrm{epi}\, f_\lambda^\iota)^{-1}(G') = (\mathrm{epi}\, f^\iota)^{-1}(G' - \mathrm{epi}\, ((\lambda p)^{-1} | \cdot |^p)).$$

Since for all ω, epi $f^\iota_\lambda(\cdot,\omega) = \mathrm{cl}(\mathrm{epi}\, f^\iota + \mathrm{epi}(\frac{1}{\lambda p}|\cdot|^p))$, to obtain the piid assertion we apply lemma 1.2 to the epigraphs. □

The distribution associated with a random lsc function is that of its epigraphical (set-valued) mapping. An alternative would be to follow the "classical" approach that associates with a random lsc function the distribution derived from (or more precisely identified by) the *marginal* distributions. For f a random function, the marginal distributions are obtained by considering all collections of random vectors that are "sections" of f, i.e., the random vectors $(f(x_1,\cdot),\dots,f(x_q,\cdot))$ for all finite collection of points $\{x_1,\dots,x_q\}$. Of course, the distribution obtained by way of the marginal distributions is not defined on the same space as that derived via the epigraphical set-valued mappings, but there could be some hope that convergence in distribution in one sense would imply the other. This is not the case! For a simple example, see [**30**, example 2.4]. But certain properties do carry over. In particular, we have the following.

Lemma 1.4. *Suppose that g^1 and g^2 are iid random lsc functions defined on $X\times\Omega$. Then, for any $x\in X$, the extended real-valued random variables $g^1(x,\cdot), g^2(x,\cdot)$ are also iid.*

Proof. Because the σ-fields induced by the random lsc functions g^1 and g^2 are independent, the independence of $g^1(x,\cdot)$ and $g^2(x,\cdot)$ would be established if we show that for $\iota = 1,2$, the σ-field induced by $g^\iota(x,\cdot)$ is contained in that generated by the random lsc function g^ι. Observe that for all $\alpha\in\mathbb{R}$, and $\varepsilon>0$,

$$A := \{\omega \,|\, g^\iota(x,\omega)\le\alpha\} \subset A_\varepsilon := \{\omega \,|\, \inf_{\mathbb{B}^o(x,\varepsilon)} g^\iota(\cdot,\omega) < \alpha+\varepsilon\},$$

where $\mathbb{B}^o(x,\varepsilon)$ denotes the open ball centered at x and of radius ε. Hence $A\subset\cap_{\varepsilon>0}A_\varepsilon$. We are claiming that equality holds. Indeed, suppose that $\omega\in A_\varepsilon$ for all $\varepsilon>0$ but ω doesn't belong to A. Then there exists x_ε with, $x_\varepsilon\to x$ as $\varepsilon\to 0$, such that for all $\varepsilon>0$, $g^\iota(x_\varepsilon,\omega)<\alpha+\varepsilon$ but $g^\iota(x,\omega)>\alpha$. From the lower semicontinuity of $x\mapsto g^\iota(x,\omega)$ it follows that

$$\lim_{\varepsilon\downarrow 0}(\alpha+\varepsilon) \ge \liminf_{\varepsilon\downarrow 0} g^\iota(x_\varepsilon,\omega) \ge g^\iota(x,\omega) > \alpha.$$

which contradicts the existence of such a ω, and thus $A=\cap_{\varepsilon>0}A_\varepsilon$. This means that sets of type A belong to the σ-field induced by the random lsc function g^i. Since the σ-field induced by the (extended real-valued) random variable $g^\iota(x,\cdot)$ can be generated by all the sets of type A, it follows that it is included in the σ-field induced by the random lsc function g^ι.

The claim that $g^1(x,\cdot)$ and $g^2(x,\cdot)$ are identically distributed, follows from the above, since for $\iota=1,2$,

$$\mu\{\omega \,|\, g^\iota(x,\omega)\le\alpha\} = \lim_{\varepsilon\downarrow 0} P_{g^\iota}(\mathcal{F}'_{G(x,\varepsilon)})$$

where $\mathcal{F}'$ refers to the closed subsets of $X\times\mathbb{R}$ and $G(x,\varepsilon) := \mathbb{B}^o(x,\varepsilon)\times(-\infty,\alpha+\varepsilon)$. □

An alternative proof for the (p)iid properties, featured in lemma 1.4 and corollary 1.3, that relies on arguments of a functional, rather than geometric, nature appears in the appendix. It exploits the properties of the Wijsman topology (cf. §2) defined on SC(X).

2. Epi-convergence.

The law of large numbers that we are interested in is formulated in terms of the *epi-convergence* of averaged sample functions. We begin with a brief review of the definitions and notations of the theory of epi-convergence, and then feature some basic facts that are going to be needed in the sequel, especially concerning topologies that induce epi-convergence.

Definition 2.1. [**5, 9**] *A sequence $\{g^\nu : X \to \overline{\mathbb{R}}, \nu \in \mathbb{N}\}$ epi-converges to g at x, if*

(a) for all $x^\nu \to x$, $\liminf_{\nu\to\infty} g^\nu(x^\nu) \geq g(x)$;

(b) there exists $\hat{x}^\nu \to x$ such that $\limsup_{\nu\to\infty} g^\nu(\hat{x}^\nu) \leq g(x)$.

If this is the case at every x in X, the $\{g^\nu\}_{\nu\in\mathbb{N}}$ epi-converge to g and one writes $g^\nu \xrightarrow{e} g$; g is called the epi-limit of the sequence $\{g^\nu\}_{\nu\in\mathbb{N}}$.

With $\mathcal{N}(x)$ the neighborhood system of x, the functions defined by

$$(\text{epi-}\liminf_\nu g^\nu)(x) := \sup_{V\in\mathcal{N}(x)} \liminf_{\nu\to\infty} \inf_{x'\in V} g^\nu(x')$$

$$(\text{epi-}\limsup_\nu g^\nu)(x) := \sup_{V\in\mathcal{N}(x)} \limsup_{\nu\to\infty} \inf_{x'\in V} g^\nu(x')$$

are the lower and upper epi-limits of the sequences $\{g^\nu\}_{\nu\in\mathbb{N}}$. When $\text{epi-}\liminf_\nu g^\nu = \text{epi-}\limsup_\nu g^\nu$, *this common function is actually the epi-limit of the sequence.*

Epi-convergence receives its name from the fact that is corresponds to the set convergence (in the Painlevé-Kuratowski sense) of the epigraphs [**5, 9, 20, 22, 27**].

Although epi-convergence is related to pointwise convergence, it is easy to see that neither epi-convergence nor pointwise convergence implies the other. However, these two notions of convergence coincide if the functions are equi-lower semicontinuous.

Definition 2.2. [**20**] *A collection $\{g^\iota, \iota \in I\} \subset$ SC(X) is equi-lower semicontinuous at x if for all $\varepsilon > 0$, $\kappa > 0$ there exists V, a neighborhood of x, such that for all $\iota \in I$,*

$$\min\left[\, f^\iota(x) - \varepsilon,\, \kappa \,\right] \leq \inf_V f^\iota.$$

If this property holds for all x in X, the collection is equi-lower semicontinuous (equi-lsc).

A sufficient condition for this property to hold is

$$g^\iota(x) < \infty \Longrightarrow g^\iota(x) - \varepsilon \leq \inf_V g^\iota$$

$$g^\iota(x) = \infty \Longrightarrow \inf_V g^\iota > \kappa.$$

Theorem 2.3. [**20**, theorem 2.6] *Let (Z, τ) be a topological space. Suppose $\{g^\nu\}_{\nu \in \mathbb{N}}$ is an equi-lsc sequence in* $\mathrm{SC}(Z)$. *Then, the sequence epi-converges to a function g if and only if it pointwise converges to g.*

In fact, if the sequence $\{g^\nu\}_{\nu \in \mathbb{N}}$ is just equi-lsc at x, then, it epi-converges at x if and only if it pointwise converges at x to the same value.

Epi-convergence can also be characterized in terms of the convergence of the epi-regularizations.

Theorem 2.4. [**7**, theorem 5.24; **5**, theorem 2.65] *Let $\{g^\nu\}_{\nu \in \mathbb{N}} \subset \mathrm{SC}(X)$ with g^ν_λ the epi-regularizations (of order p) of g^ν of index λ for some $p \in [1, \infty)$. Suppose there exist $x_0 \in X$, $\alpha_0 \geq 0$ and $\alpha_1 \in \mathbb{R}$ such that for all ν, $g^\nu \geq -\alpha_0 |\cdot - x_0|^p - \alpha_1$. Then $g^\nu \xrightarrow{e} g$ if and only if*

$$g(x) = \sup_{\lambda > 0} \limsup_{\nu \to \infty} g^\nu_\lambda(x) = \sup_{\lambda > 0} \liminf_{\nu \to \infty} g^\nu_\lambda(x).$$

Although this would take us too far afield, one can also further the view that epi-convergence is "one-sided uniform" convergence. This is confirmed by the fact that some of the basic properties of uniform convergence are shared by epi-convergence. Of particular interest in the development of almost sure convergence results is the fact that for random lsc functions when X is a separable space, epi-convergence needs only to be checked on a countable set.

Lemma 2.5. *Suppose $(X, |\cdot|)$ is a separable normed linear space and $f, g : X \to \overline{\mathbb{R}}$ with f lsc. Then there exists a countable set $D \subset X$ (that depends only on g) such that $f \leq g$ on D implies $f \leq g$ (on X). In fact, any set D obtained as the projection on X of a countable dense subset of* $\operatorname{epi} g$ *will do.*

Proof. Because $X \times \mathbb{R}$ is separable there exists a countable set $D^\uparrow$ dense in $\operatorname{epi} g$. Let D be the (canonical) projection of $D^\uparrow$ on X. Having $f \leq g$ on D is the same as requiring $\{(x, \alpha) \mid \alpha \geq g(x), x \in D\} \subset \operatorname{epi} f$. Taking closures on both sides yields $\operatorname{epi} g \subset \operatorname{epi} f$ since f is lsc, or equivalently $f \leq g$. □

Corollary 2.6. *Let $(X, |\cdot|)$ be a separable normed linear space, $\{f^\nu : X \to \overline{\mathbb{R}}, \nu \in \mathbb{N}\}$ a sequence of functions, and $f : X \to \overline{\mathbb{R}}$ a lsc function. Then, there exists a countable set $D \subset X$ (that depends only on f and on the sequence $\{f^\nu\}$) such that $f = \operatorname{epi-lim}_{\nu \to \infty} f^\nu$ on D implies $f = \operatorname{epi-lim}_{\nu \to \infty} f^\nu$ (on X).*

Proof. To show that $f = \operatorname{epi-lim}_\nu f^\nu$ we need to check the following inequalities

$$\operatorname*{epi-lim\,sup}_{\nu \to \infty} f^\nu \leq f \leq \operatorname*{epi-lim\,inf}_{\nu \to \infty} f^\nu$$

between lsc functions, see definition 2.1. The preceding lemma asserts that it suffices to have the first inequality satisfied on a certain countable set D_1, and the second one, on (possibly) another countable set D_2. We choose $D := D_1 \cup D_2$. □

A few observations about a topology related to epi-convergence might be useful to gain some insight in the functional background. For a normed linear space $(Y, |\cdot|_Y)$ and C a subset of Y, let $d(\cdot, C)$ be the distance function associated with the set C, i.e., $d(x, C) := \inf\{|x-y|_Y \mid y \in C\}$. On $\mathcal{F}_0(Y)$, the class of nonempty closed subsets of Y, the family of pseudo-distances $\{\delta_y, y \in Y\}$ defined by

$$\delta_y(C, D) := |d(y, C) - d(y, D)|, \qquad \text{for } C, D \in \mathcal{F}_0(Y)$$

induces a topology (associated with the pointwise convergence of the distance functions) called the *Wijsman topology* and denoted by τ_w; see [**22, 27, 13, 6, 14**] for more about this topology. With

$$Y = X \times \mathbb{R} \text{ and } |(x, \alpha)|_Y = \max\{|x|, |\alpha|\}$$

and by restricting our attention to $\mathcal{E}_0(Y) \subset \mathcal{F}(Y)$, the subspace of epigraphical subsets of $X \times \mathbb{R}$ that contains no "vertical" lines, the Wijsman topology on $\mathcal{E}_0(Y)$ induces a topology on $\mathrm{SC}_0(X)$, the space of proper lsc functions defined on X, by identifying a function f with its epigraph epi f. We write

$$f^\nu \xrightarrow{\tau_w} f \iff \forall\,(x, \alpha) \quad d((x, \alpha), \operatorname{epi} f^\nu) \to d((x, \alpha), \operatorname{epi} f).$$

Hess [**24, 23**] has shown that $(\mathcal{F}_0(Y), \tau_w)$ is metrizable and separable, and thus from the preceding construction it follows that $\mathrm{SC}_0(X)$ inherits the same properties. In [**24**, theorem 3.1.1], he proves that the Borel field associated with the τ_w-topology is the Effros field, which confirms the fact that this topology has a number of desirable properties when dealing with random lsc functions.

In finite dimensions, τ_w-convergence is equivalent to epi-convergence but that is not the case in general. A short proof that Wijsman convergence implies epi-convergence follows.

Proposition 2.7. *Let* $\{f; f^\nu, \nu \in \mathbb{N}\} \subset \mathrm{SC}_0(X)$. *Then*

$$f^\nu \xrightarrow{\tau_w} f \implies f^\nu \xrightarrow{e} f.$$

Proof. With $x \in \operatorname{dom} f$, $d((x, f(x)), \operatorname{epi} f) = 0$. Since $f^\nu \xrightarrow{\tau_w} f$, there exist $(x^\nu, \alpha^\nu) \to (x, f(x))$ with $\alpha^\nu \geq f^\nu(x^\nu)$, from which it follows that

$$f(x) = \lim_\nu \alpha^\nu \geq \limsup_\nu f^\nu(x^\nu).$$

This is condition (b) in definition 2.1 of epi-convergence.

Now consider a sequence $x^\nu \to x$ with $\liminf_\nu f^\nu(x^\nu) < \infty$; otherwise condition (a) of definition 2.1 is trivially satisfied. By restricting ourselves to a subsequence, if necessary, we can assume that $f^\nu(x^\nu) \to \alpha := \liminf_\nu f^\nu(x^\nu)$. From

$$|d((x^\nu, f^\nu(x^\nu)), \operatorname{epi} f^\nu) - d((x, \alpha), \operatorname{epi} f^\nu)| \leq \max\{|x^\nu - x|, |f^\nu(x^\nu) - \alpha|\},$$

it follows that $d((x,\alpha), \operatorname{epi} f^\nu) \to 0$. Since $f^\nu \xrightarrow{\tau_w} f$,

$$d((x,\alpha), \operatorname{epi} f) = 0, \ \text{ i.e., } \ \liminf_{\nu\to\infty} f^\nu(x^\nu) = \alpha \geq f(x),$$

which completes the proof. □

3. The Effros σ-algebra as a Borel structure: A variational approach

As explained briefly at the end of §2, by identifying lsc functions with their (closed) epigraphs, the Wijsman topology τ_w on $\mathrm{SC}_0(X)$ is associated with the pointwise convergence of the distance functions between epigraphs. Hess [**24, 23**] has shown that the Effros σ-algebra on $\mathrm{SC}_0(X)$ is the Borel field associated with the topology τ_w.

In this section, we identify the Effros σ-field with another Borel structure, more in step with the approach to random lsc functions featured here. We exploit the epi-regularization technology. This was the path already followed by Abdulfattah [**1**] for Moreau-Yosida regularizations.

Recall that X is a separable Banach space. We consider $\mathrm{SC}_{\alpha,p}(X)$, the subspace of lsc proper functions minorized by $-\alpha_0|\cdot - x_0|^p - \alpha_1$ for some $p \in [1,\infty)$, $\alpha_0 \geq 0$, and $\alpha_1 \in \mathbb{R}$; note that the same parameters p, α_0, α_1 apply to all functions in $\mathrm{SC}_{\alpha,p}(X)$. The epi-regularization f_λ of index λ always refers to

$$f_\lambda = f \mathbin{+_{\mathrm{e}}} \frac{1}{\lambda p} |\cdot|^p,$$

i.e., with respect to the kernel $(1/p)|\cdot|^p$; by lemma 1.1, f_λ is a real valued function, Lipschitz continuous if $p = 1$ and locally Lipschitz continuous for $p > 1$.

The family of pseudo-distances $\{\, d_{\lambda,x} \mid \lambda > 0, x \in X \,\}$ defined by

$$d_{\lambda,x}(f,g) := |f_\lambda(x) - g_\lambda(x)|$$

induces a topology on $\mathrm{SC}_{\alpha,p}(X)$ called the topology τ_{er} of pointwise convergence of the epi-regularizations with kernel $(1/p)|\cdot|^p$. Convergence with respect to τ_{er} implies epi-convergence, cf. [**5**], and like for the Wijsman topology τ_w, the converse holds when $X = \mathbb{R}^n$, but not in general.

Proposition 3.1. *The Effros σ-algebra on $\mathrm{SC}_{\alpha,p}(X)$ is the Borel field associated with the topology τ_{er}, the topology of pointwise convergence of the epigraphical regularizations.*

Proof. Let us denote by $\mathcal{S}$ the Effros field on $\mathrm{SC}_{\alpha,p}(X)$ and by $\mathcal{B}_{er}$ the Borel field associated with the topology τ_{er}.

We begin with $\mathcal{S} \subset \mathcal{B}_{er}$. By definition, we know that $\mathcal{S}$ is generated by sets of the type:

$$\{\, f \in \mathrm{SC}_{\alpha,p}(X) \mid \inf_G f > \alpha \,\}, \quad G \text{ open }, \alpha \in \mathbb{R}.$$

Observe that

$$\{ f \in \mathrm{SC}_{\alpha,p}(X) \mid \inf_G f > \alpha \} = \bigcup_{k \in \boldsymbol{N}} \{ f \in \mathrm{SC}_{\alpha,p}(X) \mid \inf_G f \geq \alpha + \frac{1}{k} \}.$$

On the other hand, since τ_{er}-convergence implies epi- convergence, it follows that ([**5**; proposition 2.9], e.g.)

$$f^\nu \xrightarrow{\tau_{er}} f \quad \Longrightarrow \quad \inf_G f \geq \limsup_\nu \inf_G f^\nu.$$

Hence $\{ f \in \mathrm{SC}_{\alpha,p}(X) \mid \inf_G f \geq \alpha + 1/k \}$ is closed with respect to the τ_{er}-topology. Thus, the set $\{ f \in \mathrm{SC}_{\alpha,p}(X) \mid \inf_G f > \alpha \}$ which is the countable union of closed sets necessarily belongs to $\mathcal{B}_{er}$.

Let us now turn to the reverse inclusion: $\mathcal{B}_{er} \subset \mathcal{S}$. For fixed λ, the epi-regularizations $\{ f_\lambda \mid f \in \mathrm{SC}_{\alpha,p}(X) \}$ are locally equi-Lipschitz continuous, and thus the τ_{er}-topology is equivalent to that induced by the pointwise convergence of these epi-regularizations on a dense subset of X. Moreover, since X is separable, it means that $\mathcal{B}_{er}$ is generated by the sets

$$\{ f \in \mathrm{SC}_{\alpha,p}(X) \;\big|\; |f_{\lambda_k}(x_k) - g_{\lambda_k}(x_k)| < \varepsilon_k \}$$

where g an arbitrary function in $\mathrm{SC}_{\alpha,p}(X)$, the (x_k) determine a countable dense subset of X, (λ_k) and (ε_k) are two sequences of strictly positive scalars that converge to zero. Let $\alpha_k := g_{\lambda_k}(x_k) - \varepsilon_k$. Taking into account

$$\{ f \in \mathrm{SC}_{\alpha,p}(X) \mid f_{\lambda_k}(x_k) \leq \alpha_k \} = \bigcap_{n \in \boldsymbol{N}} \{ f \in \mathrm{SC}_{\alpha,p}(X) \mid f_{\lambda_k}(x_k) < \alpha_k + \frac{1}{n} \},$$

to obtain $\mathcal{B}_{er} \subset \mathcal{S}$, it will thus be sufficient to show that

$$\{ f \in \mathrm{SC}_{\alpha,p}(X) \mid f_\lambda(x) < \alpha \} \in \mathcal{S}$$

for arbitrary $\lambda > 0$ and $\alpha \in \mathbb{R}$. In fact, since

$$\{ f \in \mathrm{SC}_{\alpha,p}(X) \mid f_\lambda(x) < \alpha \} = \bigcup_{n \in \boldsymbol{N}} \{ f \in \mathrm{SC}_{\alpha,p}(X) \mid f_\lambda(x) \leq \alpha - \frac{1}{n} \},$$

it will be enough to show that

$$\{ f \in \mathrm{SC}_{\alpha,p}(X) \mid f_\lambda(x) \leq \alpha \} \in \mathcal{S}$$

for arbitrary $\lambda > 0$ and $\alpha \in \mathbb{R}$. Indeed, this is equivalent to showing that $f \mapsto f_\lambda(x)$ is $\mathcal{S}$-measurable. Since f_λ is continuous and hence lsc,

$$\{ f \in \mathrm{SC}_{\alpha,p}(X) \mid f_\lambda(x) \leq \alpha \} = \bigcap_{n \in \boldsymbol{N}} \{ f \in \mathrm{SC}_{\alpha,p}(X) \mid \inf_{\mathbb{B}(x,1/n)} f_\lambda < \alpha + \frac{1}{n} \}.$$

So, we need to establish that

$$U = \{\, f \in \mathrm{SC}_{\alpha,p}(X) \mid \operatorname{epi} f_\lambda \cap \big(\operatorname{int} \mathbb{B}(x,1/n) \times (-\infty, \alpha + \frac{1}{n})\big) \neq \emptyset \,\} \in \mathcal{S}.$$

The remainder of the argument is now similar to that in lemma 1.2 and corollary 1.3. With $G = \operatorname{int} \mathbb{B}(x,1/n) \times (-\infty, \alpha + 1/n)$, an open subset of $X \times \mathbb{R}$, one has

$$\begin{aligned} U &= \{\, f \in \mathrm{SC}_{\alpha,p}(X) \mid \operatorname{cl}\big(\operatorname{epi} f + \operatorname{epi} \frac{1}{\lambda p} |\cdot|^p\big) \cap G \neq \emptyset \,\} \\ &= \{\, f \in \mathrm{SC}_{\alpha,p}(X) \mid \operatorname{epi} f \cap \big(G - \operatorname{epi} \frac{1}{\lambda p} |\cdot|^p\big) \neq \emptyset \,\}. \end{aligned}$$

Since $G - \operatorname{epi} \frac{1}{\lambda p} |\cdot|^p$ is open, we conclude that $U \in \mathcal{S}$. □

4. Basic Results.

For f a random lsc function defined on $X \times \Omega$, we reserve the notation Ef to denote the associated *expectation functional*,

$$Ef(x) := \int_\Omega f(x,\omega)\, \mu(d\omega);$$

similarly, Ef^ν is the expectation functional associated with the random lsc function f^ν. Because, the random variables $f(x,\cdot)$ are extended real-valued, we adopt the following definition for the integral: let $\Omega \to \bar{\mathbb{R}}$ be a $\mathcal{A}$-measurable function, then

$$\int h(\omega)\, \mu(d\omega) := \int h_+(\omega)\, \mu(d\omega) - \int h_-(\omega)\, \mu(d\omega)$$

where $h_+ := \max\{0, h\}$, $h_- := \max[0, -h]$ and the convention $\infty - \infty = \infty$ is to be used if the integrals of h_+ and h_- diverge; this last rule is consistent with the epigraphical context.

Lemma 4.1. *Let f be a random lsc function defined on $X \times \Omega$ with $(X, |\cdot|)$ a separable Banach space and $(\Omega, \mathcal{A}, \mu)$ a probability space. Then*

$$\omega \mapsto \inf f(\cdot,\omega) : \Omega \to \overline{\mathbb{R}}$$

is a $\mathcal{A}$-measurable function. Moreover, if, in addition, for all $\rho > 0$, $\int \inf_{\rho\mathbb{B}} f(\cdot,\omega)\, \mu(d\omega) > -\infty$ where $\rho\mathbb{B}$ is the ball of radius ρ centered at the origin, then Ef is a lsc function.

If f is minorized by a polynomial term, more precisely, there exists $p \in [1,\infty)$, $x_0 \in X$, $\alpha_0 \geq 0$ and $\alpha_1 \in \mathcal{L}^1(\Omega; \mathbb{R})$ such that for μ-almost all ω, $f(\cdot,\omega) \geq -\alpha_0 |\cdot - x_0|^p - \alpha_1(\omega)$, then for all $\lambda \in (0, (\alpha_0 p)^{-1} 2^{1-p})$, the epi-regularization of index $\lambda > 0$ (of order p) satisfies

$$f_\lambda(\cdot,\omega) = f(\cdot,\omega) +_{\mathrm{e}} \frac{1}{\lambda p} |\cdot|^p \geq -2^{p-1} \alpha_0 |\cdot - x_0|^p - \alpha_1(\omega),$$

and one has

$$Ef(x) = \lim_{\lambda \downarrow 0} Ef_\lambda(x) \quad \forall\, x \in X.$$

Proof. The measurability is an immediate consequence of the definition of a random lsc function, since that for all $\alpha \in R$, the set

$$\{\omega \mid \inf f(\cdot, \omega) < \alpha\} = \{\omega \mid \operatorname{epi} f(\cdot, \omega) \cap X \times (-\infty, \alpha) \neq \emptyset\} \in \mathcal{A}.$$

Keeping in mind that every sequence $x^\nu \to x$ must eventually lie in a ball of radius ρ, the definition of the integral and the assumption that $\int \inf_{\rho \mathbf{B}} f(\cdot, \omega)\, \mu(d\omega) > -\infty$ allows us to apply Fatou's lemma (for extended real-valued functions) and conclude that Ef is lsc. The same arguments as those in corollary 1.3 imply that for every $\lambda > 0$, f_λ is a random lsc function. The minorization $f_\lambda(\cdot, \omega) \geq -2^{p-1}\alpha_0|\cdot - x_0|^p - \alpha_1(\omega)$ is immediate since $|\cdot|^p \geq 0$. After observing (lemma 1.1(a)) that the collection $\{f_\lambda, \lambda > 0\}$ is monotone increasing as λ goes to 0, a direct appeal to the monotone convergence theorem (applied to any countable sequence $\lambda_\nu \downarrow 0$) completes the proof. □

As far as the pointwise convergence of averaged random lsc functions is concerned, we have the following.

Lemma 4.2. *Let X be a separable Banach space and $\{f^\nu : X \times \Omega \to (-\infty, \infty],\, \nu \in \mathbb{N}\}$ piid random lsc functions. Suppose that for all $\rho \geq 0$, $\int \inf_{\rho \mathbf{B}} f^1(\cdot, \omega)\, \mu(d\omega) > -\infty$ where $\rho\, \mathbb{B}$ is the ball of radius ρ centered at the origin. Then, Ef^1 is lsc, and for all $x \in X$*

$$Ef^1(x) = \lim_{\nu \to \infty} \frac{1}{\nu} \sum_{k=1}^{\nu} f^k(x, \omega) \quad \mu\text{-a.s.}.$$

Proof. We already know by lemma 4.1 that Ef^1 is lsc. Etemadi's version of the standard law of large numbers [**21**], (with a trivial extension to allows for random variables with values in $(-\infty, \infty]$) applied to the piid (lemma 1.4) random variables $\{f^\nu(x, \cdot), \nu \in \mathbb{N}\}$ yields the second assertion. □

We are now set to state the main result about the epi-convergence of an arbitrary collection of random lsc functions.

Theorem 4.3. *Suppose X is a separable Banach space and $\{f^\nu : X \times \Omega \to \overline{\mathbb{R}}, \nu \in \mathbb{N}\}$ is a sequence of piid random lsc functions bounded below μ-almost surely by a polynomial minorant: $f^\nu(x, \omega) \geq -\alpha_0|x - x_0|^p - \alpha_1(\omega)$ with $p \in [1, \infty)$, $x_0 \in X$, $\alpha_0 \in \mathbb{R}_+$, and $\alpha_1 \in \mathcal{L}^1(\Omega; \mathbb{R})$ Then, for $\mu-$almost all ω,*

$$Ef^1 = \operatorname*{epi-lim}_{\nu \to \infty} \frac{1}{\nu} \sum_{k=1}^{\nu} f^k(\cdot, \omega).$$

Proof. Let $g^\nu(x, \omega) := \frac{1}{\nu} \sum_{k=1}^{\nu} f^k(x, \omega)$. The functions g^ν are also random lsc functions. From the standard strong law of large numbers, via lemma 4.2, we know that for all $x \in X$,

$Ef^1(x) = \lim_\nu g^\nu(x,\omega)$ μ-a.s., in particular $\limsup_\nu g^\nu(x,\omega) = Ef^1(x)$ for all $\omega \in \Omega_x$ with $\mu(\Omega_x) = 1$. It follows directly from the definition 2.1 of epi-limits that $\text{epi-}\limsup_\nu g^\nu \le \limsup g^\nu$, and thus for all $x \in X$,

$$\mathop{\text{epi-}\limsup}_{\nu\to\infty} g^\nu(x,\omega) \le Ef^1(x), \quad \forall\, \omega \in \Omega_x;$$

definition 2.1 also implies that for all ω, the functions $x \mapsto \text{epi-}\limsup_\nu g^\nu(x,\omega)$ are lsc. Also Ef^1 is lsc as follows from lemma 4.1. Let D be the (canonical) projection of $D^\uparrow$ a countable dense subset of $\operatorname{epi} Ef^1 \subset X \times \mathbb{R}$. The existence of such a set is guaranteed by the separability of X. Let $\Omega_e = \cap_{x\in D}\Omega_x$. Because D is countable, Ω_e is of μ-measure 1, and for all x in D and all ω in Ω_e, we have

$$(\mathop{\text{epi-}\limsup}_{\nu\to\infty} g^\nu(\cdot,\omega))(x) \le Ef^1(x).$$

We now appeal to lemma 2.5 to conclude that for all $\omega \in \Omega_e$,

$$\mathop{\text{epi-}\limsup}_{\nu\to\infty} g^\nu(\cdot,\omega) \le Ef^1.$$

There remains to show that $Ef^1 \le \text{epi-}\liminf_\nu g^\nu(\cdot,\omega)$ μ-a.s..

Let's begin with a proof of the inequality, under the additional assumption that the random lsc functions f^ν are μ-almost surely equi-bounded above, i.e., there exists θ such that for μ-almost all ω, $f^\nu(\cdot,\omega) \le \theta$. Let's define

$$-\alpha_1^\nu(\omega) := \inf_x \,[\, f^\nu(x,\omega) + \alpha_0|x - x_0|^p \,].$$

These are are piid random variables:

(i) the functions being minimized are random lsc functions and consequently, for all ν, $\alpha_1^\nu : \Omega \to \overline{\mathbb{R}}$ is $\mathcal{A}$-measurable (lemma 4.1);

(ii) since the functions $(x,\omega) \mapsto f^\nu(x,\omega) + \alpha_0|x - x_0|^p$ are piid random lsc functions (corollary 1.3);

the piid properties of the α_1^ν now following directly from the observation:

$$\{\, \omega \mid \alpha_1^\nu(\omega) < \eta \,\} = \{\, \omega \mid \inf f^\nu(x,\omega) + \alpha_0|x - x_0|^p < \eta \,\}.$$

Simply observe that the sets on the right belong to the independent sigma-fields induced by the f^ν, and the sets on the left determine a generating class for the distribution of the α_1^ν and they carry the same probability mass for all ν since the f^ν are identically distributed. This identity also implies that for all ν, the α_1^ν are independent from f^ℓ whenever $\ell \ne \nu$. Moreover, since μ-almost surely

$$-\alpha_1 \le -\alpha_1^\nu \le f^\nu(x_0,\cdot) + |x_0 - x_0|^p \le \theta,$$

one can appeal to the Etemadi's version of the law of large numbers to obtain

$$\sum_{k=1}^{\nu} \alpha_1^k(\omega) \to E\{\alpha^1(\cdot)\} =: \bar{\alpha}_1 \in \mathbb{R} \quad \text{for } \mu\text{-almost all } \omega \ .$$

For $\nu \in \mathbb{N}$, let $j^\nu(x,\omega) := f^\nu(x,\omega) + \alpha_1^\nu(\omega) \geq -\alpha_0|x - x_0|^p$, and observe that they are random lsc functions (as the sum of two random lsc functions). Moreover, this collection $\{j^\nu : X \times \Xi \to \overline{\mathbb{R}},\ \nu \in \mathbb{N}\}$ of random lsc functions is piid. To see this, simply observe that for any open set G and $\alpha \in \mathbb{R}$, the sets

$$\{\,\omega \mid \inf_G j^\nu(\cdot,\omega) < \alpha\,\} = \{\,\omega \mid \inf_G f^\nu(\cdot,\omega) < \alpha - \inf f^\nu(\cdot,\omega)\,\} \in \mathcal{A}_{f^\nu},$$

determine a generating class for the distribution of the j^ν. If the inequality $Ef^1 \leq$ epi-lim $\inf_\nu g^\nu(\cdot,\omega)$ μ-a.s. is known to hold when $\alpha^1 \equiv 0$, applying this to the j^ν, one would be able to conclude that for μ-almost all ω,

$$Ef^1 + \bar{\alpha}_1 = Ej^1 \leq \operatorname*{epi\text{-}lim\,inf}_{\nu\to\infty} \frac{1}{\nu}\sum_{k=1}^{\nu} j^k(\cdot,\omega) = \operatorname*{epi\text{-}lim\,inf}_{\nu\to\infty} \left(\frac{1}{\nu}\sum_{k=1}^{\nu} f^k(\cdot,\omega) + \frac{1}{\nu}\sum_{k=1}^{\nu} \alpha_1^\nu(\omega)\right).$$

Since, for μ-almost all ω, $\frac{1}{\nu}\sum_{k=1}^{\nu} \alpha_1^\nu(\omega) \to \bar{\alpha}_1$, and since epi-convergence is preserved under the addition of scalars converging to a finite limit, for μ-almost all ω,

$$\operatorname*{epi\text{-}lim\,inf}_{\nu\to\infty} \left(\frac{1}{\nu}\sum_{k=1}^{\nu} f^k(\cdot,\omega) + \frac{1}{\nu}\sum_{k=1}^{\nu} \alpha_1(\omega)\right) = \operatorname*{epi\text{-}lim\,inf}_{\nu\to\infty} \frac{1}{\nu}\sum_{k=1}^{\nu} f^k(\cdot,\omega) \ + \ \bar{\alpha}_1.$$

And, hence one would also have $Ef^1 \leq$ epi-lim $\inf_\nu \frac{1}{\nu}\sum_{k=1}^{\nu} f^k(\cdot,\omega)$. So, let's proceed with $\alpha_1 \equiv 0$.

The next step is to show that the inequality holds when the functions f^ν are replaced by their epi-regularizations. For $\lambda > 0$, let $f_\lambda^\nu(\cdot,\omega)$ be the epi-regularizations of $f^\nu(\cdot,\omega)$ of index λ (order p), i.e.,

$$f_\lambda^\nu(\cdot,\omega) = f^\nu(\cdot,\omega) +_{\mathrm{e}} (\lambda p)^{-1}|\cdot|^p.$$

In view of corollary 1.3, for all $\lambda > 0$, $\{f_\lambda^\nu\}_{\nu\in\mathbb{N}}$ is a sequence of piid random lsc functions. Moreover, for λ sufficiently close to zero, the functions $x \mapsto f_\lambda^\nu(x,\omega)$ are μ-almost surely real-valued and equi-locally Lipschitzian (lemma 1.1); the equi-boundedness assumption gets used here to guarantee that the same (local) Lipschitz constants will be available. Let $h^\nu(x,\omega;\lambda) := (1/\nu)\sum_{k=1}^{\nu} f_\lambda^k(x,\omega)$. For all $\lambda \in (0,\lambda_0)$ ($\lambda_0 \leq (\alpha_0 p)^{-1}2^{1-p}$) and all $\omega \in \Omega_0$ for a certain set Ω_0 of μ-measure 1, all the functions $x \mapsto h^\nu(x,\omega;\lambda)$ are real-valued and equi-locally Lipschitz (with the same Lipschitz constants as for $f_\lambda^\nu(\cdot,\omega)$). Consequently, for all $\lambda \in (0,\lambda_0)$, $\omega \in \Omega_0$, the functions $x \mapsto \lim_\nu h^\nu(x,\omega;\lambda)$ are locally Lipschitz. The Etemadi's version of the law of large numbers for real-valued random variables, via lemma

4.2, implies that for all $\lambda \in (0, \lambda_0)$, to every $x \in X$ there corresponds a set $\Omega_{x,\lambda} \subset \Omega_0$ of μ-measure 1 such that

$$\begin{aligned} Ef^1_\lambda(x) &= \lim_{\nu\to\infty} h^\nu(x,\omega\,;\lambda) \quad \forall \omega \in \Omega_{x,\lambda} \\ &= \operatorname{epi-lim}_{\nu\to\infty} h^\nu(\cdot,\omega\,;\lambda)(x) \quad \forall \omega \in \Omega_{x,\lambda} \end{aligned}$$

where the second equality follows from theorem 2.3. Let D be a countable dense subset of X and $\Omega_\lambda = \cap_{x\in D}\Omega_{x,\lambda}$; then $\mu(\Omega_\lambda) = 1$. For every $\lambda \in (0, \lambda_0)$ and every $\omega \in \Omega_\lambda$:

$$Ef^1_\lambda = \operatorname{epi-lim}_{\nu\to\infty} h^\nu(\cdot,\omega;\lambda) = \operatorname{epi-lim\,inf}_{\nu\to\infty} h^\nu(\cdot,\omega;\lambda) \text{ on } D,$$

$$x \mapsto \operatorname{epi-lim\,inf}_{\nu\to\infty} h^\nu(x,\omega;\lambda) \text{ is locally Lipschitz },$$

$$D \times \mathbb{Q} \cap \operatorname{epi}(\operatorname{epi-lim\,inf}_\nu) h^\nu(\cdot,\omega;\lambda) \text{ is dense in } \operatorname{epi}(\operatorname{epi-lim\,inf}_\nu) h^\nu(\cdot,\omega;\lambda).$$

We now appeal to lemma 2.5 to conclude that for all $\lambda \in (0, \lambda_0)$,

$$Ef^1_\lambda \le \operatorname{epi-lim\,inf}_{\nu\to\infty} h^\nu(\cdot,\omega;\lambda), \quad \forall \omega \in \Omega_\lambda.$$

Since for all $\lambda > 0$, $f^\nu_\lambda(\cdot,\omega) \le f^\nu(\cdot,\omega)$, this also implies that for all $\lambda \in (0, \lambda_0)$,

$$Ef^1_\lambda \le \operatorname{epi-lim\,inf}_{\nu\to\infty} \frac{1}{\nu}\sum_{k=1}^{\nu} f^k(\cdot,\omega) \quad \forall \omega \in \Omega_\lambda.$$

Now, choose a sequence $\lambda_l \downarrow 0$ and observe that $\Omega_e = \cap_{l\in\mathbb{N}}\Omega_{\lambda_l}$ has μ- measure 1. By letting $l \to \infty$ and relying on lemma 4.1, we obtain $Ef^1 \le \operatorname{epi-lim\,inf}_\nu (1/\nu)\sum_{k=1}^{\nu} f^k$ μ-almost surely (for all $\omega \in \Omega_e$).

There remains only to consider the general case when the f^ν are not necessarily equi-bounded μ-almost surely. For $\theta \in \mathbb{R}$, let $f^\nu_{[\theta]}$ be the (upper) truncation of f^ν, i.e.,

$$f^\nu_{[\theta]}(x,\omega) := \min\{\theta, f^\nu(x,\omega)\}.$$

Because $\operatorname{epi} f^\nu_{[\theta]}(\cdot,\omega) = \operatorname{epi} f^\nu(\cdot,\omega) \cup \{(x,\alpha) \,|\, \alpha \ge \theta\}$, it is immediate that the $f^\nu_{[\theta]}$ are random lsc functions. Moreover, for any θ, the random functions $f^\nu_{[\theta]}$ are piid. To see that, let $E^\nu := \operatorname{epi} f^\nu$ and $E^\nu_{[\theta]} := \operatorname{epi} f^\nu_{[\theta]}$ be the epigraphical set-valued mappings from Ω into the (epigraphical) subsets of $X \times \mathbb{R}$, let $H_\theta := \{(x,\alpha) \,|\, \alpha \ge \theta\}$, and note that for any open set $G \subset X \times \mathbb{R}$,

$$(E^\nu_{[\theta]})^{-1}(G) = \begin{cases} (E^\nu)^{-1}(G), & \text{if } G \cap H_\theta = \emptyset; \\ \Omega & \text{if } G \cap H_\theta \ne \emptyset. \end{cases}$$

The sigma-fields generated by the closed random sets $E^\nu_{[\theta]}$ are independent since those generated by the random sets E^ν are independent by assumption. Similarly, for the distributions $P^\nu_{[\theta]}$ associated with the random closed $E^\nu_{[\theta]}$, for G open,

$$P^\nu_{[\theta]}(\mathcal{F}_G) = \begin{cases} P^\nu(\mathcal{F}_G) & \text{if } G \cap H_\theta = \emptyset; \\ 1 & \text{if } G \cap H_\theta \ne \emptyset; \end{cases}$$

where P^ν is the distribution of E^ν. This shows that the $E^\nu_{[\theta]}$ are identically distributed since $\{\mathcal{F}_G,\ G \text{ open }\}$, as already observed in lemma 1.2, is a generating class for the (Effros) field $\mathcal{S}$.

The inequality obtained in the case of μ-a.s. equi-bounded random lsc functions shows that for all $\theta \in \mathbb{R}$, there exists $\Omega_\theta \subset \Omega$ of μ-measure 1 such that for all $\omega \in \Omega_\theta$,

$$Ef^1_{[\theta]} \leq \operatorname*{epi-lim\,inf}_{\nu\to\infty}(1/\nu)\sum_{k=1}^{\nu} f^k_{[\theta]}(\cdot,\omega) \leq \operatorname*{epi-lim\,inf}_{\nu\to\infty}(1/\nu)\sum_{k=1}^{\nu} f^k(\cdot,\omega),$$

where the last inequality follows from $f^k_{[\theta]} \leq f^k$. Now pick $\{\theta^l\}_{l=1}^\infty$ a sequence that goes to ∞, and let $\Omega' = \cap_l \Omega_{\theta_l}$. The preceding inequality holds for all θ_l and all $\omega \in \Omega'$ where $\mu(\Omega') = 1$. As $l \to \infty$, the sequence $Ef^1_{[\theta_l]}$ converges to Ef^1 (Beppo Levi's monotone convergence theorem), and thus for all $\omega \in \Omega'$,

$$Ef^1 = \lim_{l\to\infty} Ef^1_{[\theta_l]} \leq \operatorname*{epi-lim\,inf}_{\nu\to\infty}(1/\nu)\sum_{k=1}^{\nu} f^k(\cdot,\omega),$$

which completes the proof. □

The theorem can be used to obtain an almost sure pointwise convergence result for a large class of random lsc functions identified by the following definition.

Definition 4.4 [30]. *A collection of random lsc functions $\{f^\iota : X \times \Omega \to \overline{\mathbb{R}},\ \iota \in I\}$ is μ-almost surely equi-lower semicontinuous if there exists Ω_1 with $\mu(\Omega_1) = 1$ such that the collection of functions $\{f^\iota(\cdot,\omega),\ \iota \in I, \omega \in \Omega_1\}$ is equi-lsc.*

Corollary 4.5. *Let X be a separable Banach space, $(\Omega, \mathcal{A}, \mu)$ a probability space with $\mathcal{A}$ μ-complete, $\{f^\nu : X \times \Omega \to \overline{\mathbb{R}},\ \nu \in \mathbb{N}\}$ piid random lsc functions bounded below μ-almost surely by a polynomial minorant, as in theorem 4.3, and μ-a.s. equi-lsc. Then, μ-almost surely*

$$Ef^1 = \lim_{\nu\to\infty} \frac{1}{\nu}\sum_{k=1}^{\nu} f^k(\cdot,\omega) = \operatorname*{epi-lim}_{\nu\to\infty} \frac{1}{\nu}\sum_{k=1}^{\nu} f^k(\cdot,\omega).$$

Proof. This is a immediate consequence of the preceding definition and theorem 2.3, using the fact that the functions $x \mapsto \nu^{-1}\sum_{k=1}^{\nu} f^k(x,\omega)$ are also equi-lsc (the infimum of a sum is always larger than or equal to the sum of the infima). □

Remark 4.6. We note that the assumptions of theorem 4.3 are not sufficient to yield μ-almost pointwise-convergence, the equi-lsc condition is needed. Lemma 4.2 *does not* imply that the averaged random lsc functions converge pointwise to Ef^1 μ-a.s., since $\frac{1}{\nu}\sum_{k=1}^{\nu} f^k(x,\omega) \to Ef^1(x)$ may fail for a different set $\Omega \setminus \Omega_x$ (of measure 0) and $\cap_x \Omega_x$ may very well not be of measure 1. □

5. Strong law of large numbers and the epi-distance

Here again, $(X, |\cdot|)$ will be a separable Banach space, $\mathbb{B}$ is the unit ball (centered at the origin) and as in section 2, $d(x, C) := \inf_{y \in C} |x - y|$ is the distance between a point x and a set C; $d(x, \emptyset) = \infty$. For $C, D \subset X$, possibly empty, and $\rho \geq 0$, let $C_\rho := C \cap \rho\,\mathbb{B}$, and the *excess* of C_ρ on D is defined as

$$e_\rho(C, D) := \sup\{\, d(x, D) \mid x \in C_\rho \,\}.$$

The *ρ-set distance* between $C, D \subset X$ is

$$\hat{d\!l}_\rho(C, D) = \max\{\, e_\rho(C, D), e_\rho(D, C) \,\}.$$

For two functions $f, g : X \to \overline{\mathbb{R}}$, the *epi-distance* of index ρ is

$$\hat{d\!l}_\rho(f, g) = \hat{d\!l}_\rho(\operatorname{epi} f, \operatorname{epi} g)$$

where $\operatorname{epi} f, \operatorname{epi} g \subset X \times \mathbb{R}$, and $X \times \mathbb{R}$ is given the "sup" norm: $|(x, \alpha)| := \max\{\, |x|, |\alpha| \,\}$. A sequence of sets $\{\, C^\nu \subset X, \nu \in \mathbb{N} \,\}$ converges to a set C with respect to the ρ-set distance if for all $\rho > 0$, $\hat{d\!l}_\rho(C^\nu, C) \to 0$ as $\nu \to \infty$; the topology τ_{aw} engendered by this convergence notion has been analyzed in [**10, 6, 12, 17**]. Similarly, the functions $\{\, f^\nu, \nu \in \mathbb{N} \,\}$ converge to a function f with respect to the epi-distance if $\hat{d\!l}_\rho(f^\nu, f) \to 0$ as $\nu \to \infty$ for all $\rho > 0$. We denote by τ_{aw} the topology generated by the epi-distance on $\mathrm{SC}(X)$, and by $\xrightarrow{aw}$ convergence with respect to the epi-distance topology.

Estimates for the epi-distance are provided by the following criterion.

Kenmochi Conditions 5.1. [**10**, Theorem 2.1] *Suppose f, g are proper extended real-valued functions defined on a normed linear space X, both minorized by $-\alpha_0 |\cdot|^p - \alpha_1$ for some $\alpha_0 \geq 0$, $\alpha_1 \in \mathbb{R}$ and $p \geq 1$. Let $\rho_0 > 0$ be such that $(\operatorname{epi} f)_{\rho_0}$ and $(\operatorname{epi} g)_{\rho_0}$ are nonempty.*

a) *Then the following conditions hold: for all $\rho > \rho_0$ and $x \in \operatorname{dom} f$ such that $|x| \leq \rho$, $|f(x)| \leq \rho$, for every $\varepsilon > 0$ there exists some $\tilde{x}_\varepsilon \in \operatorname{dom} g$ that satisfies*

$$|x - \tilde{x}_\varepsilon| \leq \hat{d\!l}_\rho(f, g) + \varepsilon$$
$$g(\tilde{x}_\varepsilon) \leq f(x) + \hat{d\!l}_\rho(f, g) + \varepsilon$$

as well as a symmetric condition that interchanges the roles of f and g.

b) *Conversely, assuming that for all $\rho > \rho_0 > 0$ there exists a "constant" $\eta(\rho) \in \mathbb{R}_+$, depending on ρ, such that for all $x \in \operatorname{dom} f$ with $|x| \leq \rho, |f(x)| \leq \rho$, there exists $\tilde{x} \in \operatorname{dom} g$ that satisfies*

$$|x - \tilde{x}| \leq \eta(\rho),$$
$$g(\tilde{x}) \leq f(x) + \eta(\rho),$$

and the symmetric condition (interchanging the roles of f and g), then with $\rho_1 := \rho + \alpha_0 \rho^p + |\alpha_1|$,

$$\hat{d\!l}_\rho(f, g) \leq \eta(\rho_1).$$

This epi-distance topology τ_{aw} can also be built from the pseudo-distances $d_{\theta,\rho}$ introduced in [**8**], based on the uniform convergence of the Moreau-Yosida approximates on bounded sets: for $p, q \in \mathrm{SC}_0(X)$,

$$d_{\theta,\rho}(p,q) := \sup_{|x|\leq\rho} |p_\theta(x) - q_\theta(x)|, \quad \theta > 0, \rho \geq 0.$$

It has been shown that $\{\hat{d\!\!l}_\rho, \rho \geq 0\}$ and $\{d_{\theta,\rho}, \theta > 0, \rho \geq 0\}$ induce the same uniform structure on $\mathrm{SC}_0(X)$ [**6**]. Thus we may as well work with the latter when dealing with convergence with respect to the epi-distance. Moreover, when the functions $f^\nu(\cdot,\omega)$ are convex, we can event restrict our attention to $d_{1,\rho}$ since for any sequence of proper, convex functions $\{q^\nu\}_{\nu\in\mathbb{N}} \subset \mathrm{SC}_0(X)$, $d_{\theta,\rho}(q^\nu, q) \to 0$ for all $\theta > 0, \rho \geq 0$ if and only if $d_{1,\rho}(q^\nu, q) \to 0$ for all $\rho > 0$ [**10, 8**].

We work with X a separable Banach space, $(\Omega, \mathcal{A}, \mu)$ a probability space with $\mathcal{A}$ μ-complete, and f and $\{f^\nu, \nu \in \mathbb{N}\}$ random lsc functions defined on $X \times \Omega$. We are going to assume that the $f^\nu(\cdot,\omega)$ belong to a separable subspace of $(\mathrm{SC}(X), \tau_{aw})$. This would certainly be the case if X is finite dimensional [**6**, theorem 5.1], and also if the $f^\nu(\cdot,\omega)$ belong to a τ_{aw}-compact subset of $\mathrm{SC}(X)$; for τ_{aw}-compactness criteria refer to §3 of [**6**], in particular [**6**, theorems 3.2, 3.3]. Conditions (c) and (d) in theorem 5.2 assume that the pseudo-distances $d_{\theta,\rho}$ between the sums of the functions f^k and their Moreau-Yosida regularizations converge to 0 as $\lambda{\downarrow}0$.

Theorem 5.2. *Let X be a separable Banach space, $\{ f^\nu : X \times \Omega \to \overline{\mathbb{R}}, \nu \in \mathbb{N} \}$ a sequence of random lsc proper functions, i.e., for all $\nu \in \mathbb{N}$, $\omega \in \Omega$, $f^\nu(\cdot,\omega) \in \mathrm{SC}_0(X)$. Assume that*

(a) for all ν, $f^\nu(\cdot,\omega)$ belong to a separable subspace of $(\mathrm{SC}_0(X), \tau_{aw})$;
(b) the random lsc functions f^ν are piid;
(c) for all $\theta > 0$, $\rho \geq 0$, $\nu \in \mathbb{N}$ and for almost all $\omega \in \Omega$:

$$d_{\theta,\rho}\big(\nu^{-1}\sum_{k=1}^{\nu} f^k(\cdot,\omega),\, \nu^{-1}\sum_{k=1}^{\nu} f^k_\lambda(\cdot,\omega)\big) \leq \varepsilon_{\theta,\rho}(\lambda) \quad \text{with} \quad \varepsilon_{\theta,\rho}(\lambda) \to 0 \;\text{ as }\; \lambda{\downarrow}0;$$

(d) for all $\theta > 0$, $\rho \geq 0$: $d_{\theta,\rho}(Ef^1_\lambda, Ef^1){\downarrow}0$ as $\lambda{\downarrow}0$.

Then

$$\frac{1}{\nu}\sum_{k=1}^{\nu} f^k \xrightarrow{aw} Ef^1 \qquad \mu\text{-a.s.}\ .$$

Proof. Let

$$g^\nu := \frac{1}{\nu}(f^1 + \cdots + f^\nu), \qquad g^\nu_{\lambda+} := \frac{1}{\nu}(f^1_\lambda + \cdots + f^\nu_\lambda).$$

Both g^ν and $g^\nu_{\lambda+}$ are random lsc functions, cf §1.

From the triangle inequality for $d_{\theta,\rho}$, for all $\lambda > 0$ and μ-almost all ω, one has

$$\begin{aligned} d_{\theta,\rho}(g^\nu(\cdot,\omega), Ef^1) &\leq d_{\theta,\rho}(g^\nu(\cdot,\omega), g^\nu_{\lambda+}(\cdot,\omega)) + d_{\theta,\rho}(g^\nu_{\lambda+}(\cdot,\omega), Ef^1_\lambda) + d_{\theta,\rho}(Ef^1_\lambda, Ef^1), \\ &\leq \varepsilon_{\theta,\rho}(\lambda) + d_{\theta,\rho}(g^\nu_{\lambda+}(\cdot,\omega), Ef^1_\lambda) + d_{\theta,\rho}(Ef^1_\lambda, Ef^1). \end{aligned}$$

Let us consider the embedding

$$h \mapsto (h_\lambda)_{|\rho\,\mathbb{B}} : (\mathrm{SC}_0(X), \tau_{aw}) \to (\mathcal{C}(\rho\,\mathbb{B}), |\cdot|_{\sup}),$$

where $h_{|\rho\,\mathbb{B}}$ means the restriction of the functions h to the ρ-ball, $\mathcal{C}(\rho\,\mathbb{B})$ is the space of continuous functions on the ρ-ball, and $|\cdot|_{\sup}$ is the sup-norm. The embedding is τ_{aw}-continuous. For fixed $\lambda > 0$, the random lsc functions $(\omega \mapsto f^\nu_\lambda(\cdot,\omega))$ are piid (corollary 1.3). Moreover, on separable subsets of $\mathrm{SC}_0(X)$, the Effros and the Borel fields associated with the epi-distance topology coincide [**11** , theorem 4.4]. Hence, by continuity of the embedding $h \mapsto (h_\lambda)_{|\rho\,\mathbb{B}}$, the vector-valued random variables $(f^\nu_\lambda)_{|\rho\,\mathbb{B}}$ are also piid. Moreover, since the $f^\nu(\cdot,\omega)$ belong to a separable subspace of $\mathrm{SC}_0(X)$, the functions $(f^\nu_\lambda)_{|\rho\,\mathbb{B}}$ also belongs to separable subspace of $\mathcal{C}(\rho\,\mathbb{B})$, again by the continuity of the embedding. Now, let us fix a sequence $\lambda_k \downarrow 0$. From the law of large numbers for vector-valued random variables [**21**], for all k there exists a set $\Omega_k \subset \Omega$ of μ-measure 1 such that for all $\omega \in \Omega_k$

$$g^\nu_{\lambda_k+} \to Ef^1_{\lambda_k} \text{ uniformly on } \rho\,\mathbb{B},\ \forall \rho \geq 0.$$

Thus, with $\Omega_e = \cap_k \Omega_k$ –note that $\mu(\Omega_e) = 1$– we have that for all $\theta > 0, \rho > 0$ and all $\omega \in \Omega_e$,

$$d_{\theta,\rho}(g^\nu_{\lambda_k+}(\cdot,\omega), Ef^1_{\lambda_k}) \to 0 \text{ as } \nu \to \infty.$$

Hence, for all $\theta > 0$, $\rho > 0$ and all $\omega \in \Omega_e$:

$$\limsup_{\nu\to\infty} d_{\theta,\rho}(g^\nu(\cdot,\omega), Ef^1) \leq \varepsilon_{\theta,\rho}(\lambda) + d_{\theta,\rho}(Ef^1_\lambda, Ef^1).$$

Letting $\lambda \downarrow 0$, and appealing to assumptions (c) and (d), one concludes that

$$\limsup_{\nu\to\infty} d_{\theta,\rho}(g^\nu(\cdot,\omega), Ef^1) \leq 0,$$

which in turn implies that $g^\nu(\cdot,\omega) \xrightarrow{aw} Ef^1$ μ-a.s.. □

6. The convex case: Mosco epi-convergence

In the convex case it is possible to obtain a stronger version of theorem 4.3. It will be shown that the Ef^1 is the epi-limit of the sample means $\frac{1}{\nu}\sum_{k=1}^{\nu} f^k(\cdot,\omega)$ with respect to *both* the strong and the weak topologies.

We assume that $(X, |\cdot|)$ is a reflexive Banach space. We know, from the John-Ziezler theorem [**19**, p. 185] (an extension of the Kadeč-Klee theorem) that X admits an equivalent norm so that both X, and its dual X^*, are locally uniformly convex. We assume that $|\cdot|$ is such a smooth norm. We rely on this smoothness of the norm in the proof which is based on approximation techniques involving Moreau-Yosida regularizations, cf. [**5**, section 2.7].

Definition 6.1. *Let X be a reflexive Banach space. A sequence $\{h^\nu : X \to \overline{\mathbb{R}}, \nu \in \mathbb{N}\}$ Mosco epi-converges to h at x, if*

(a) *for all $x^\nu \underset{w}{\to} x$ (converging weakly), $\liminf_{\nu\to\infty} h^\nu(x^\nu) \geq h(x)$;*

(b) *there exists $\hat{x}^\nu \to x$ (converging strongly) such that $\limsup_{\nu\to\infty} h^\nu(\hat{x}^\nu) \leq h(x)$.*

If this is the case at every x in X, the sequence $\{h^\nu\}_{\nu\in\mathbb{N}}$ Mosco epi-converges to h and we write $h^\nu \xrightarrow{M\text{:}e} h$.

Mosco epi-convergence corresponds to the set convergence of the epigraphs but now with respect to both the weak and the strong topology, cf. [**28, 7, 5, 20**]. We work with $\mathrm{SCC}_0(X) = \{h : X \to \overline{\mathbb{R}},\ \text{lsc, convex, proper}\}$. The Legendre-Fenchel transform on $\mathrm{SCC}_0(X)$ is the one-to-one mapping:

$$h \to h^* : \mathrm{SCC}_0(X) \to \mathrm{SCC}_0(X^*), \quad h^*(v) = \sup_{x\in X}\{\langle x, v\rangle - h(x)\}$$

where $\langle\cdot,\cdot\rangle$ is the bilinear form that brings X and X^* in duality; moreover one has $h = (h^*)^*$. The set of *subgradients* of the convex function h at x is the set

$$\partial h(x) := \{v \in X^* | h(x') \geq h(x) + \langle x' - x, v\rangle, \quad \forall\, x' \in X\}.$$

The *resolvant of index* λ of ∂h, is the operator J_λ^h. For all x, $J_\lambda^h(x)$ is the unique point that minimizes $h + \frac{1}{2\lambda}|x - \cdot|^2$, i.e.,

$$J_\lambda^h x := \operatorname*{argmin}_{u\in X}\{\, h(u) + \frac{1}{2\lambda}|x - u|^2 \,\}$$

(one refers also to J_λ^h as the *prox map*) and thus for the Moreau-Yosida regularization h_λ, we have

$$h_\lambda(x) = h(J_\lambda^h x) + \frac{1}{2\lambda}|x - J_\lambda^h x|^2.$$

We know that $x \to J_\lambda^h x : X \to X$ is continuous and everywhere defined; in the Hilbert case this is a contraction [**15**]. Because $J_\lambda^h x$ is a minimum point

$$0 \in \partial h(J_\lambda^h x) + \frac{1}{\lambda} H(J_\lambda^h x - x)$$

with $H := \partial(\frac{1}{2}|\cdot|^2)$ the duality map. The Yosida approximate of ∂h, denoted $(\partial h)_\lambda$, is defined by

$$(\partial h)_\lambda(x) = \frac{1}{\lambda} H(x - J_\lambda^h x) = \nabla h_\lambda(x),$$

since h_λ is Fréchet differentiable, in fact C^1; see [**5, 15**] for more about Moreau-Yosida regularizations.

Theorem 6.2. *Suppose $(X, |\cdot|)$ is a separable reflexive Banach space with $|\cdot|$ a smooth norm, $(\Omega, \mathcal{A}, \mu)$ a probability space, $\{f^\nu : X \times \Omega \to \overline{\mathbb{R}}, \nu \in \mathbb{N}\}$ a sequence of piid random lsc functions such that for μ-almost all ω, the functions $f^\nu(\cdot, \omega) \in \mathrm{SCC}_0(X)$. Suppose moreover that $\mu-$almost surely, the $f^\nu(\cdot, \omega)$ are bounded below by the quadratic minorant $f^\nu(x, \omega) \geq -\alpha_0|x - x_0|^2 - \alpha_1(\omega)$ with $x_0 \in X$, $\alpha_0 \in \mathbb{R}_+$, $\alpha_1 \in \mathcal{L}^1(\Omega; \mathbb{R})$ and suppose there exists $\hat{u} \in \mathcal{L}^2(\Omega; X)$ such that $\int f^1(\hat{u}(\omega), \omega)\, \mu(d\omega) < \infty$. Then, μ-almost surely*

$$\frac{1}{\nu} \sum_{k=1}^{\nu} f^k(\cdot, \omega) \xrightarrow{M:e} Ef^1.$$

Proof. Let $g^\nu := \frac{1}{\nu} \sum_{k=1}^{\nu} f^k$. In view of theorem 4.3 and its proof, one needs only to show that there exists a set $\Omega_M \in \mathcal{A}$ of μ-measure 1 such that for all $\omega \in \Omega_M$ and for all $x^\nu \underset{w}{\to} x$ (converging weakly),

$$\liminf_{\nu \to \infty} g^\nu(x^\nu, \omega) \geq Ef^1(x).$$

Let f_λ^ν be the Moreau-Yosida regularization (with respect to the first variable) of index $0 < \lambda < (4\alpha_0)^{-1}$. These are piid random lsc functions (corollary 1.3) and equi-locally Lipschitz continuous. We first observe [**5**, theorem 3.26] that because the norm $|\cdot|$ is smooth (by assumption), for functions $h^\nu, h \in \mathrm{SCC}_0(X)$, one has

$$h^\nu \xrightarrow{M:e} h \iff h_\lambda^\nu(x) \to h_\lambda(x) \quad \forall\, x \in X,\ \forall\, \lambda > 0$$

i.e., $h^\nu \xrightarrow{M:e} h \iff h^\nu \xrightarrow{\tau_{er}} h$; cf. §3 for the definition of τ_{er} and recall that by proposition 3.1, the Borel field associated with τ_{er} is precisely the Effros σ-algebra on $\mathrm{SCC}_0(X)$. Again by [**5**, theorem 3.26] one has

$$h^\nu \xrightarrow{M:e} h \iff \nabla h_\lambda^\nu(x) \to \nabla h_\lambda(x) \quad \forall\, x \in X,\ \forall\, \lambda > 0.$$

Thus,

$$h \mapsto \nabla h_\lambda(x) : \mathrm{SCC}_0(X) \to X^*$$

is continuous with respect to τ_{er}, and hence, the vector-valued random variables

$$\{\, \nabla f_\lambda^\nu(x, \omega),\ \nu \in \mathbb{N}\,\} \qquad \text{for } \lambda > 0, x \in X$$

are piid. Moreover, by lemma 4.1 and the assumptions of the theorem, for all $x \in X$:

$$-2\alpha_0|x - x_0|^2 - \alpha_1(\omega) \leq f_\lambda^\nu(x, \omega) \leq f^\nu(\hat{u}(\omega), \omega) + \tfrac{1}{2}|x - \hat{u}(\omega)|^2$$

which implies that the integral functional

$$u \mapsto I_{f_\lambda^\nu}(u) = \int_\Omega f_\lambda^\nu(u(\omega), \omega)\, \mu(d\omega)$$

is a finite-valued convex continuous function on $\mathcal{L}^2(\Omega)$. From the theory of convex integral functionals [**29**] and the C^1-continuity of f^ν_λ, it follows that for every $u \in \mathcal{L}^2(\Omega)$,

$$\nabla I_{f^\nu_\lambda}(u) = \nabla f^\nu_\lambda(u(\cdot),\cdot)$$

which belongs to $\mathcal{L}^2(\Omega)$. In particular, for every $x \in X$, $\nabla f^\nu_\lambda(x,\cdot)$ belongs to $\mathcal{L}^2(\Omega)$ and hence to $\mathcal{L}^1(\Omega)$. We can apply Etemadi's version of the strong law of large numbers for vector-valued random variables [**21**] to conclude that there exists a set $\Omega_e \subset \Omega$ of μ-measure 1 such that for all $\omega \in \Omega_e$ and all $x \in D$, a dense countable subset of X,

$$\frac{1}{\nu}\sum_{k=1}^{\nu} \nabla f^\nu_\lambda(x,\omega) \;\to\; E\{\nabla f^1_\lambda(x,\cdot)\}.$$

Since the functions $f^\nu(\cdot,\omega)$ and $g^\nu(\cdot,\omega)$ are convex, for all $\omega \in \Omega_e$ and all y in D,

$$g^\nu(x^\nu,\omega) \geq \frac{1}{\nu}\sum_{k=1}^{\nu} f^k_\lambda(x^\nu,\omega) \geq \frac{1}{\nu}\sum_{k=1}^{\nu} f^k_\lambda(y,\omega) + \frac{1}{\nu}\langle \sum_{k=1}^{\nu} \nabla f^k_\lambda(y,\omega), x^\nu - y\rangle.$$

Taking liminf on both sides, we see that the last term converges to $\langle E\{\nabla f^1_\lambda(y,\cdot)\}, x-y\rangle$ since we are dealing with the product of strongly and weakly convergent sequences. And, thus for all $\omega \in \Omega_e$,

$$\liminf_{\nu\to\infty} g^\nu(x^\nu,\omega) \geq Ef^1_\lambda(y) + \langle E\{\nabla f^1_\lambda(y,\cdot)\}, x-y\rangle;$$

recall that for λ sufficiently small, Ef^1_λ is finite valued. Now letting $y \to x$, and exploiting the continuity of $y \mapsto \nabla f^1_\lambda(y,\cdot)$ and $y \mapsto Ef^1_\lambda(y)$, we conclude that

$$\liminf_{\nu\to\infty} g^\nu(x^\nu,\omega) \geq Ef^1_\lambda(x).$$

Finally, we let $\lambda \downarrow 0$ to obtain $\liminf_\nu g^\nu(x^\nu,\omega) \geq Ef^1(x)$ using lemma 4.1. □

Our next result is concerned with a "dual" version of theorem 6.2 that is directly related to the law of large numbers for random sets of Artstein, Hart and Vitale [**3, 2**], Hess [**23**] and Hiai [**25**].

Lemma 6.3. *Let X be a reflexive Banach space and h^ν, h convex, proper, real-valued functions defined on X. Then $(h^\nu)^* \xrightarrow{M:e} h^*$ implies $h^\nu \xrightarrow{M:e} h^{**} = \operatorname{cl} h$.*

Proof. Since $(h^\nu)^* \xrightarrow{M:e} h^*$, it follows from the continuity of the Legendre-Fenchel transform with respect to the Mosco epi-convergence [**28, 5**], that $(h^\nu)^{**} \xrightarrow{M:e} h^{**}$. Thus for every $x^\nu \underset{w}{\to} x$,

$$\liminf_{\nu\to\infty} h^\nu(x^\nu) \geq \liminf_{\nu\to\infty} (h^\nu)^{**}(x^\nu) \geq h^{**}(x).$$

On the other hand, for proper convex functions, $(h^\nu)^{**} = \operatorname{cl} h^\nu$. Hence, $\text{epi-lim}\, h^\nu = \text{epi-lim}(h^\nu)^{**} = h^{**}$ which guarantees that for all x there exists $x^\nu \to x$ (convergence with respect to the norm) such that

$$h^{**}(x) \geq \limsup_{\nu\to\infty} h^\nu(x^\nu).$$

Thus, both conditions (a) and (b) of definition 6.1 are satisfied. □

The next result refers to the epi-integral of a random lsc function. Let $\oint$ denote the *epi-integral* (or "continuous inf-convolution" as introduced by Valadier [**31**]). For $f : X \to \overline{\mathbb{R}}$,

$$\begin{aligned}(\oint f\, d\mu)(x) &= [\oint f(\cdot,\omega)\,\mu(d\omega)](x)\\ &:= \inf_{u\in\mathcal{U}}\{\int_\Omega f(u(\omega),\omega)\,\mu(d\omega) \,\Big|\, \int_\Omega u(\omega)\,\mu(d\omega) = x\},\end{aligned}$$

where $\mathcal{U} = \{\, u : \Omega \to X \,\big|\, u\ \mathcal{A}\text{-summable}\}$.

Proposition 6.4. *Suppose X is a separable reflexive Banach space, $(\Omega, \mathcal{A}, \mu)$ a probability space, $\{f^\nu : X \times \Omega \to \overline{\mathbb{R}},\ \nu \in \mathbb{N}\}$ a sequence of piid random lsc function, such that for μ-almost all ω, the functions $x \mapsto f^\nu(x,\omega) \in \mathrm{SCC}_0(X)$. Suppose moreover that μ-almost surely the $(f^\nu)^*(\cdot,\omega)$ are bounded below by the quadratic minorant $-\alpha_0|\cdot - v_0|_*^2 - \alpha_1(\omega)$ for some $\alpha_0 \in \mathbb{R}_+,\ \alpha_1 \in \mathcal{L}^1(\Omega;\mathbb{R}),\ v_0 \in X^*$, and suppose there exists $v \in \mathcal{L}^2(\Omega; X^*)$ such that $\int (f^1)^*(v(\omega),\omega)\,\mu(d\omega) < \infty$. Then, μ-almost surely*

$$(1/\nu) *_{\mathrm{e}} [f^1(\cdot,\omega) +_{\mathrm{e}} \cdots +_{\mathrm{e}} f^\nu(\cdot,\omega)] \xrightarrow{M:e} \operatorname{cl} \oint f^1(\cdot,\omega)\,\mu(d\omega).$$

Proof. Since the Legendre-Fenchel transform $f \mapsto f^*$ is continuous with respect to Mosco epi-convergence and the random lsc functions f^ν are piid, it follows that also the random lsc functions $(f^\nu)^*$ are piid. We apply theorem 6.2 to obtain that μ-a.s.,

$$\frac{1}{\nu}[(f^1)^* + \cdots + (f^\nu)^*] \xrightarrow{M:e} (E(f^1)^*).$$

Now, observe that

$$\left(\frac{1}{\nu} *_{\mathrm{e}} [f^1 +_{\mathrm{e}} \cdots +_{\mathrm{e}} f^\nu]\right)^* = \frac{1}{\nu}[(f^1)^* + \cdots + (f^\nu)^*],$$

and apply lemma 6.3 to obtain

$$\frac{1}{\nu} *_{\mathrm{e}} [f^1(\cdot,\omega) +_{\mathrm{e}} \cdots +_{\mathrm{e}} f^\nu(\cdot,\omega)] \xrightarrow{M:e} (E(f^1)^*)^*.$$

There remains only to appeal to the results in [**16**, theorem VIII.40] about the conjugates of epi-integrals to obtain

$$(E(f^1)^*)^*(\cdot) = \operatorname{cl} \oint f^1(\cdot,\omega)\,\mu(d\omega),$$

and this completes the proof. ◻

Again with X a separable reflexive Banach space, let

$$\{\, \Gamma^\nu : \Omega \rightrightarrows X, \quad \nu \in \mathbb{N} \,\}$$

be a collection of piid random closed sets such that

$$E\Gamma^1 := \{x \in X \mid x = \int u\, d\mu, u \text{ summable selection of } \Gamma^1\} \neq \emptyset.$$

The law of large numbers of Artstein-Vitale and Hess-Hiai asserts that μ-a.s.,

$$\frac{1}{\nu}[\Gamma^1 + \cdots + \Gamma^\nu] \xrightarrow{M} \operatorname{cl}\operatorname{con} E\Gamma^1.$$

Proposition 6.4 is a consequence of this law of large of large numbers applied to the random sets $\Gamma^\nu(\omega) = \operatorname{epi} f^\nu(\cdot,\omega)$, simply notice that

$$\operatorname{cl}(1/\nu)[\operatorname{epi} f^1(\cdot,\omega) + \cdots + \operatorname{epi} f^\nu(\cdot,\omega)] = \operatorname{cl}\operatorname{epi}(1/\nu) *_{\text{e}} [f^1(\cdot,\omega) +_{\text{e}} \cdots +_{\text{e}} f^\nu(\cdot,\omega)].$$

Once proposition 6.4 has been derived in this way, one can proceed to theorem 6.2 by making use of the bicontinuity of the Legendre-Fenchel transform as in lemma 6.3. This was the path followed in [**26**]. On the other hand, one can also obtain the law of large numbers for piid random *convex* sets as a consequence of proposition 6.4. This is almost immediate under the assumptions that the Γ^ν are piid, and that Γ^1 admits a $\mathcal{L}^2$-selection, i.e., there exists $\hat{x} : \Omega \to X$, $\hat{x} \in \mathcal{L}^2(\Omega; X)$ such that $\hat{x}(\omega) \in \Gamma^1(\omega)$ for μ-almost all ω. One simply applies proposition 6.4 to the collection of random lsc functions

$$\{\, (x,\omega) \mapsto \delta_{\Gamma^\nu(\omega)}(x),\ \nu \in \mathbb{N} \,\},$$

i.e., the indicator functions of the random sets Γ^ν, whose conjugates are

$$\{\, (v,\omega) \mapsto \sigma_{\Gamma^\nu(\omega)}(v),\ \nu \in \mathbb{N} \,\},$$

the support functions associated with these random sets. From the existence of $\mathcal{L}^2$-selections, it follows that

$$\sigma_{\Gamma^1(\omega)}(v) \geq \langle v, \hat{x}(\omega)\rangle \geq -2(|v|_*^2 + |\hat{x}(\omega)|^2);$$

one makes use of the inequality $\langle v, \hat{x}(\omega)\rangle \leq |v|_*\, |\hat{x}(\omega)| \leq 2(|v|_*^2 + |\hat{x}(\omega)|^2)$. Choosing $\alpha_0 = 2$, $v_0 = 0$, $\alpha_1(\cdot) = 2|\hat{x}(\cdot)|^2$, which belongs to $\mathcal{L}^1(\Omega; \mathbb{R})$, yields the quadratic minorant.

This approach only yields the law of large numbers for piid convex random sets. In the general case, a convexification argument needs to be introduced as done in [**2, 23, 25**].

Appendix. Iid properties: functional proof

Here we develop another argument, of a functional nature, that yields the iid properties of random lsc functions established in section 1. In particular, we use the fact that υ_x, the valuation mapping, is measurable. More precisely, when $\mathrm{SC}(X)$ is equipped with the Wijsman topology, the function $\upsilon_x : f \mapsto f(x)$ is lsc, whereas $\upsilon_{x,\lambda} : f \mapsto f_\lambda(x)$ is usc (upper semicontinuous) and thus both are Borel measurable. To see that $\upsilon_{x,\lambda}$ is usc, it suffices to observe that for all $\varepsilon > 0$, there exists u_x such that

$$f(u_x) + \frac{1}{2\lambda}\|u_x - x\|^2 - \frac{\varepsilon}{2} \leq f_\lambda(x) \leq f(u_x) + \frac{1}{2\lambda}\|u_x - x\|^2.$$

Now choose $\eta > 0$ such that $\varepsilon\lambda \leq \eta^2 + 2\eta(\|u_x - x\| + 1)$. Then for every g with $d((u_x, f(u_x)), \operatorname{epi} g) \leq \eta$, we have

$$g_\lambda(x) \leq f(u_x) + \eta + \frac{1}{2\lambda}[\eta^2 + 2\eta\|u_x - x\| + \|u_x - x\|^2] \leq f_\lambda(x) + \varepsilon.$$

Proposition A.1. *Let f, g be epi-iid random lsc functions. Then for all $x \in X$ and $\lambda > 0$*

(a) $f(x, \cdot)$ and $g(x, \cdot)$ are iid extended real-valued random variables.

If in addition, as a function of x, f and g are minorized by a quadratic form and λ is close enough to 0, then

(b) $f_\lambda(x, \cdot)$ and $g_\lambda(x, \cdot)$ are iid real-valued random variables.

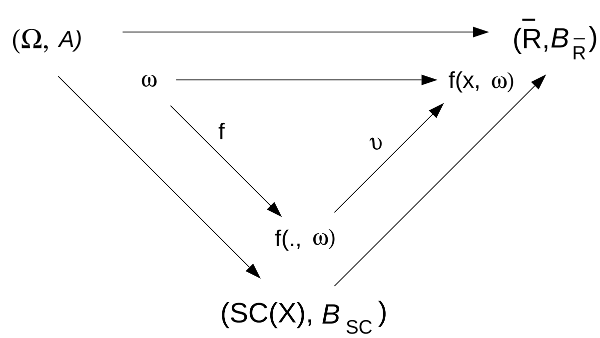

Fig. A.. The random function f and the random variable $f(x, \cdot)$

Proof. We rely on the diagram in Fig. A. Observe that $\omega \mapsto f(x, \omega) = \upsilon_x \circ f$. Hence

$$(\upsilon_x \circ f)^{-1}(\mathcal{B}_{\overline{\mathbb{R}}}) = f^{-1}(\upsilon_x^{-1}(\mathcal{B}_{\overline{\mathbb{R}}})) \subset \mathcal{A}_f$$

since υ_x is a measurable function (i.e., $\upsilon_x^{-1}(\mathcal{B}_{\overline{\mathbb{R}}}) \subset \mathcal{B}_X$). When $\mathcal{A}_f$ and $\mathcal{A}_g$ are independent, so are $(\upsilon_x \circ f)^{-1}(\mathcal{B}_{\overline{\mathbb{R}}}) \subset \mathcal{A}_f$ and $(\upsilon_x \circ g)^{-1}(\mathcal{B}_{\overline{\mathbb{R}}}) \subset \mathcal{A}_g$. The same type of argument (utilizing this time the measurability of the valuation map $\upsilon_{x,\lambda}$) also works for $f_\lambda(\cdot, x)$ and $g_\lambda(\cdot, x)$. This proves independence.

Moreover, for any $B \in \mathcal{B}_{\overline{\mathbb{R}}}$,

$$\begin{aligned} \mu\{\omega \,|\, f(x,\omega) \in B\} &= \mu\{\omega \,|\, (\upsilon_x \circ f)(\omega) \in B\} \\ &= \mu\{\omega \,|\, f(\cdot,\omega) \in (\upsilon_x)^{-1}(B)\} \\ &= \mu\{\omega \,|\, g(\cdot,\omega) \in (\upsilon_x)^{-1}(B)\} \\ &= \mu\{\omega \,|\, g(x,\omega) \in B\} \end{aligned}.$$

A similar argument shows that f_λ and g_λ are identically distributed. □

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