RANDOM LSC FUNCTIONS: SCALARIZATION *

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Abstract. Random lsc (lower semicontinuous) functions can be identified with a vector-valued random variable by means of an appropriate scalarization. It is shown that stationarity, ergodicity and independence properties are preserved by this scalarization. The scalarization is exploited to obtain an lsc version of the conditional expectation of a random lsc function.

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1. Introduction

Let lsc-fcns(X) denote the space of lsc (lower semicontinuous) extended real-valued functions defined on a Polish (complete, separable, metric) space (X,d), and $(\Xi,\mathcal{S},P)$ a probability space. By the *scalarization* of a random lsc function $f:\Xi\to$ lsc-fcns(X), we mean the identification of f with a countable collection of random variables, say $\{\pi^k:\Xi\to\overline{I\!R},\ k\in I\!N\}$, that uniquely defines f. We bring to the fore a particular class of scalarizations that preserve ergodicity properties. Such scalarizations are exploited in [5] to derive ergodic limit laws for random lsc functions and here, in §4, to construct an lsc version of the conditional expectation of a random lsc function.

2. Probabilistic framework

We adapt the standard probabilistic framework to lsc-fcns(X)-valued random variables. A set-valued mapping $S:\Xi\rightrightarrows X$ is a *random set* if it is *measurable*, i.e.,

$$\text{for any open set } O\subset X,\quad \{\xi\in\Xi\,|\,S(\xi)\cap O\neq\emptyset\}=:S^{-1}(O)\in\mathcal{S}.$$

It's a *random closed set* if in addition it is *closed-valued*: for all $\xi\in\Xi$, $S(\xi)$ is closed; note that the definition implies that $\{\xi\in\Xi\,|\,S(\xi)\neq\emptyset\}=S^{-1}(X)=:\operatorname{dom}S\in\mathcal{S}$.

A *random lsc (lower semicontinuous) function* is a function $f:\Xi\to$ lsc-fcns(X) such that the associated *epigraphical mapping*

$$S_f:\Xi\rightrightarrows X\times I\!R\quad\text{with}\quad S_f(\xi):=\operatorname{epi}f(\xi)=\{(x,\alpha)\in X\times I\!R\,|\,\alpha\geq f(\xi)(x)\}$$

is a random closed set. We find it convenient to identify an lsc function $f(\xi)$ with its bivariate representation

$$(\xi,x)\mapsto f(\xi,x):\Xi\times X\to\overline{I\!R},$$

so we usually write $f(\xi,x)$ instead of $f(\xi)(x)$ for the value of $f(\xi)$ at x. For all $\xi\in\Xi$, the function $x\mapsto f(\xi,x)$ is then lsc. Recall that a function $g:X\to\overline{I\!R}$ is *lsc at* $\bar{x}$ if for all $x^\nu\to\bar{x}$, $\liminf_\nu g(x^\nu)\geq g(\bar{x})$. Equivalently, g is lsc at $\bar{x}$ if for some (decreasing) sequence $\{V^\nu\subset X,\ \nu\in I\!N\}$ of neighborhoods of $\bar{x}$ such that $V^\nu\supset V^{\nu+1}$ and $\bigcap_{\nu\in N}V^\nu=\{\bar{x}\}$,

$$p^\nu(\bar{x}):=\inf_{x\in V^\nu}g(x)\nearrow g(\bar{x}).$$

Further, g is *lsc* if and only if $\operatorname{epi}g\subset X\times I\!R$ is closed. The concept of a random lsc function is due to Rockafellar [6] who introduced it in the context of the calculus of variations under the name of 'normal integrand'; see Chapter 14 of [7] for a systematic exposition.

The *Effrös field* identifies the σ-field of measurable subsets of lsc-fcns(X). It's the σ-field generated by the sets of the type

$$\mathcal{A}_{D,\alpha}=\{f\in\text{lsc-fcns}(X)\,|\,\inf_D f\leq\alpha\},$$

with D either an open or closed subset of X; cf. [3] [5, §4], for more about the properties of the Effrös field on lsc-fcns(X).

To a random lsc function f one associates its *distribution* P_f defined by

$$P_f(\mathcal{A}) := P(\{\xi \in \Xi \mid f(\xi, \cdot) \in \mathcal{A}\}) \quad \text{for } \mathcal{A} \in \mathcal{E}.$$

Two random lsc functions, f and g, are *identically distributed* if for all $\mathcal{A} \in \mathcal{E}$, $P_f(\mathcal{A}) = P_g(\mathcal{A})$. The *joint distribution* of a finite collection $\{f^1, \ldots, f^k\}$ of random lsc functions is given, for $\mathcal{A}_1, \ldots, \mathcal{A}_k \in \mathcal{E}$, by

$$P_{f^1,\ldots,f^k}(\mathcal{A}_1, \ldots, \mathcal{A}_k) := P(\{\xi \in \Xi \mid f^1(\xi, \cdot) \in \mathcal{A}_1, \ldots, f^k(\xi, \cdot) \in \mathcal{A}_k\}).$$

For a sequence $\{f^\nu,\ \nu \in I\!N\}$ of random lsc functions, let's denote by P^∞ the probability measure on the sequence space $(\text{lsc-fcns}(X)^\infty, \mathcal{E}^\infty)$ that is consistent with the joint distribution of the f^ν. That such a measure exists follows from Kolmogorov's extension theorem for random lsc functions, cf. [5, §4].

Random lsc functions are said to be *independent* if their distributions are independent. A sequence of random lsc functions $\{f^\nu,\ \nu \in I\!N\}$ of random lsc functions is *pairwise independent* if for any pair $k, l \in I\!N$ and $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{E}$,

$$P_{f^k,f^l}(\mathcal{A}_1, \mathcal{A}_2) = P_{f^k}(\mathcal{A}_1) P_{f^l}(\mathcal{A}_2).$$

The sequence is *independent* if for any finite subcollection, $\{f^{\nu_1}, \ldots, f^{\nu_k},\ k \in I\!N\}$,

$$P_{f^{\nu_1},\ldots,f^{\nu_k}}(\mathcal{A}_1, \ldots, \mathcal{A}_k) = \Pi_{i=1}^k P_{f^{\nu_i}}(\mathcal{A}_i) \text{ for any sets } \mathcal{A}_1, \ldots, \mathcal{A}_k \in \mathcal{E}.$$

Definition 2.1 (iid and stationarity)**.** *A sequence, $\{f^\nu,\ \nu \in I\!N\}$ of random lsc functions is iid (independent and identically distributed) if it is independent and for any $k, l \in I\!N$, f^k and f^l are identically distributed. The sequence is stationary if its joint distributions are invariant under shifts in the sequence, more precisely, for any finite subcollection $\{f^{\nu_1}, \ldots, f^{\nu_k}\}$, $k \in I\!N$, any $l \in I\!N$ and any $\mathcal{A}_1, \ldots, \mathcal{A}_k \in \mathcal{E}$, one has*

$$P_{f^{\nu_1},\ldots,f^{\nu_k}}(\mathcal{A}_1, \ldots, \mathcal{A}_k) = P_{f^{\nu_1+l},\ldots,f^{\nu_k+l}}(\mathcal{A}_1, \ldots, \mathcal{A}_k).$$

Stationarity can also be characterized in terms of a measure preserving transformation. Recall that a function $\varphi : \Xi \to \Xi$ is *measure preserving* if for all $A \in \mathcal{S}$, $P(\varphi^{-1}(A)) = P(A)$. If f is a random lsc function, one verifies easily that the sequence $\{f, f\circ\varphi, f\circ\varphi^2, \ldots\}$ is stationary. In fact, every stationary sequence of random lsc functions can be redefined in terms of a (single) random lsc function and a measure preserving transformation:
Say $\{f^\nu,\ \nu \in I\!N\}$ is a stationary sequence of random lsc functions and P^∞ the measure induced on $(\text{lsc-fcns}(X)^\infty, \mathcal{E}^\infty)$. Redefine the f^ν as follows:

$$f^\nu : \text{lsc-fcns}(X)^\infty \to \text{lsc-fcns}(X) \text{ with } f^\nu(\zeta) := \zeta^\nu,$$

i.e., the ν-th element of the sequence $\zeta = (\zeta^1, \zeta^2, \dots) \in \text{lsc-fcns}(X)^\infty$. The new sequence $\{f^\nu,\ \nu \in I\!N\}$ is stationary and has the same joint distributions as the original one, but now with respect to a new probability space. Letting $\varphi : \text{lsc-fcns}(X)^\infty \to \text{lsc-fcns}(X)^\infty$ be the shift operator,

$$\varphi(\zeta^1, \zeta^2, \dots) := (\zeta^2, \zeta^3, \dots),$$

and defining $f : \text{lsc-fcns}(X)^\infty \to \text{lsc-fcns}(X)$ as $f(\zeta) = \zeta^1$, one has $f(\varphi^\nu(\zeta)) = \zeta^{\nu+1}$, so that $f, f\circ\varphi, f\circ\varphi^2, \dots$, defines the same stationary sequence on $\text{lsc-fcns}(X)^\infty$ with respect to the measure preserving shift transformation φ; it is easy to check that φ is measure preserving.

If $\varphi : \Xi \to \Xi$ is measure preserving, then $A \in \mathcal{S}$ is an *invariant event* if $\varphi^{-1}(A) = A$ almost surely, i.e., in terms of the symmetric difference, $P(\varphi^{-1}(A) \triangle A) = 0$.

Definition 2.2 (ergodicity)**.** *Let $\mathcal{I} \subset \mathcal{E}$ denote the σ-field of invariant events and call it the invariant σ-field. A measure preserving map $\varphi : \Xi \to \Xi$ is ergodic if $\mathcal{I}$ is trivial, i.e., for all $A \in \mathcal{I}$, $P(A) \in \{0, 1\}$. A sequence $\{f^\nu,\ \nu \in I\!N\}$ of random lsc functions is ergodic if the associated (measure preserving) shift operator φ on the sequence space $(\text{lsc-fcns}(X)^\infty, \mathcal{E}^\infty, P^\infty)$ is ergodic.*

3. Scalarization of random lsc functions

The framework of reference is still that of §1-2: (X, d) Polish space with $\mathcal{B}$ the associated Borel field on X, and $(\Xi, \mathcal{S}, P)$ a probability space. It will now be shown that in this setting, a random lsc function f is completely identified by a countable collection of extended real-valued random variables

$$f \;\longleftrightarrow\; \{\pi_{x,\rho} \,|\, x \in R, \rho \in Q\!\!\!\!\!\;\,{}_+\} \text{ where } R \text{ is a countable dense subset of } X.$$

As in §2, let's begin with some well-known properties of random sets [1, 4]. Propositions 3.2 and 3.3 follow directly from the definitions, X Polish, and a not so surprising, but nontrivial, projection result:

Theorem 3.1 (measurable projection theorem), [1, Theorem III.23]**.** *Suppose $\mathcal{S}$ is P-complete and G is an $\mathcal{S} \otimes \mathcal{B}$-measurable subset of $\Xi \times X$. Then, $\text{prj}_\Xi G \in \mathcal{S}$, i.e., the projection of G on Ξ is $\mathcal{S}$ measurable.*

Proposition 3.2. *Suppose $S : \Xi \rightrightarrows X$ is a random set and for all $\xi \in \Xi$, let $\text{cl}\, S(\xi) = \text{cl}\,(S(\xi))$. Then $\text{cl}\, S : \Xi \rightrightarrows X$ is a random closed set.*

Proof. For $O \subset X$ open, clearly $O \cap S(\xi) \neq \emptyset$ if and only if $O \cap \text{cl}\, S(\xi) \neq \emptyset$. □

Proposition 3.3. *A closed-valued mapping $S : \Xi \rightrightarrows X$ is a random closed set if and only if $S^{-1}(D) \in \mathcal{S}$ for all $D \in \mathcal{D}$ where $\mathcal{D}$ is any one of the following collections of subsets of X:*

(a) $\mathcal{D}$ = *the open balls* $\mathbb{B}^o(x,\rho) = \{x' \in X \mid d(x',x) < \rho\}$;

(b) $\mathcal{D}$ = *the open rational balls* $\mathbb{B}^o(x,\rho)$ *with* $x \in R$, *a dense countable subset of* X, *and* $\rho \in \mathbb{Q}_+$.

Moreover, if X *is* σ*-compact or if* $\mathcal{S}$ *is* P*-complete, then* $\mathcal{D}$ *can also be any one of the following collection of subsets of* X:

(c) $\mathcal{D}$ = *the closed sets* C;

(d) $\mathcal{D}$ = *the closed balls* $\mathbb{B}(x,\rho)$;

(e) $\mathcal{D}$ = *the closed rational balls* $\mathbb{B}(x,\rho)$ *with* $x \in R$, *a dense countable subset of* X, *and* $\rho \in \mathbb{Q}_+$.

Proof. Clearly, S measurable $\Rightarrow$(a)$\Rightarrow$(b). Since X is Polish, every open set O can be written as the countable union of open rational balls: with $x^\nu \in R$ and $\rho^\nu \in \mathbb{Q}_+$, one has

$$O = \bigcup_{\nu=1}^{\infty} \mathbb{B}^o(x^\nu,\rho^\nu), \qquad S^{-1}(O) = \bigcup_{\nu=1}^{\infty} S^{-1}(\mathbb{B}^o(x^\nu,\rho^\nu)) \in \mathcal{S}.$$

Also, (c)$\Rightarrow$(d)$\Rightarrow$(e) doesn't need proof. An argument similar to the one above yields (e)$\Rightarrow$ S measurable.

Now let's assume that X is σ-compact and show that S measurable $\Rightarrow$ (c). Every closed set $C \subset X$ can now be written as the countable union of compact sets $\{B^\nu,\ \nu \in \mathbb{N}\}$ from which follows that $S^{-1}(C) = \bigcup_{\nu \in \mathbb{N}} S^{-1}(B^\nu)$. It now suffices to observe that for all $\nu \in \mathbb{N}$, $S^{-1}(B^\nu) \in \mathcal{S}$. Indeed, even with X just a metric space, given any nonempty, compact set $D \subset X$, define the open sets $D^\nu := \{x \in X \mid d(x,D) < 1/\nu\}$ for $\nu \in \mathbb{N}$. Since $S(t) \cap D \neq \emptyset$ if and only if $S(t) \cap D^\nu \neq \emptyset$ for all $\nu \in \mathbb{N}$, we have $S^{-1}(D) = \bigcap_\nu S^{-1}(D^\nu)$. Hence, $S^{-1}(D)$ is the intersection of a countable collection of measurable sets and therefore is itself measurable.

Finally, let's assume that $\mathcal{S}$ is P-complete and again show that S measurable $\Rightarrow$ (c). Let R be a countable dense subsets of X. Since S is closed-valued, $\bar{x} \in S(\xi)$ if and only if for all $\rho \in \mathbb{Q}_+$, there exists $x_\rho \in R$ such that $\bar{x} \in \mathbb{B}^o(x_\rho,\rho)$ and $S(\xi) \cap \mathbb{B}^o(x_\rho,\rho) \neq \emptyset$. This means that

$$\begin{aligned} \operatorname{gph} S &= \{(\xi,x) \in \Xi \times X \mid x \in S(\xi)\} \\ &= \textstyle\bigcap_{\rho \in \mathbb{Q}_+} \bigcup_{x \in R} \left[S^{-1}(\mathbb{B}^o(x,\rho)) \times \mathbb{B}^o(x,\rho)\right]. \end{aligned}$$

is a $\mathcal{S} \otimes \mathcal{B}$-measurable subset of $\Xi \times X$, where $\mathcal{B}$ is the Borel field on (X,d). Indeed, by (b), each set $S^{-1}(\mathbb{B}^o(x,\rho)) \times \mathbb{B}^o(x,\rho)$ belongs to $\mathcal{S} \times \mathcal{B}$ and $\operatorname{gph} S$ can be written as the countable intersection of a countable union of sets of this type. To complete the proof, one appeals to the Projection Theorem 3.1 which yields

$$S^{-1}(C) = \operatorname{prj}_\Xi(\operatorname{gph} S \cap (\Xi \times C)) \in \mathcal{S}$$

since $\operatorname{gph} S \cap (\Xi \times C) \in \mathcal{S} \otimes \mathcal{B}$. □

Theorem 3.4 (scalarization of random lsc functions)**.** *Let* $f : \Xi \to \text{lsc-fcns}(X)$

$$\text{and for } D \subset X : \qquad \text{let } \pi_D(\xi) := \inf_{x \in D} f(\xi, x).$$

Then, f *is a random lsc function if and only if for all* $D \in \mathcal{D}$, π_D *is measurable where* $\mathcal{D}$ *is any one of the following collection of subsets of* X*:*

(a) $\mathcal{D}$ = *the open sets* O*;*

(b) $\mathcal{D}$ = *the open balls* $I\!B^o(x, \rho) = \{x' \in X \mid d(x', x) < \rho\}$*;*

(c) $\mathcal{D}$ = *the open rational balls* $I\!B^o(x, \rho)$ *with* $x \in R$*, a dense countable subset of* X*, and* $\rho \in Q_+$*.*

Moreover, if X *is* σ*-compact or* $\mathcal{S}$ *is complete, then* $\mathcal{D}$ *can also be any one of the following collection of subsets of* X*:*

(d) $\mathcal{D}$ = *the closed sets* C*;*

(e) $\mathcal{D}$ = *the closed balls* $I\!B(x, \rho)$*;*

(f) $\mathcal{D}$ = *the closed rational balls* $I\!B(x, \rho)$ *with* $x \in R$ *and* $\rho \in Q_+$*.*

Proof. Let (m) stand for the property: f is a random lsc function. We show that

$$\text{(m)} \Longrightarrow \text{(a)} \Longrightarrow \text{(b)} \Longrightarrow \text{(c)} \Longrightarrow \text{(m)}.$$

Only the implications (m) $\Longrightarrow$ (a) and (c) $\Longrightarrow$ (m) need proof.

(m) $\Longrightarrow$ (a): For any open set $O \subset X$ and $\alpha \in I\!R$,

$$\begin{aligned}\pi_O^{-1}((-\infty, \alpha)) &= \{\xi \in \Xi \mid \inf_O f(\xi, \cdot) < \alpha\} \\ &= \{\xi \in \Xi \mid S_f(\xi) \cap (O \times (-\infty, \alpha)) \neq \emptyset\} = S_f^{-1}(O \times (-\infty, \alpha)) \in \mathcal{S}\end{aligned}$$

since $O \times (-\infty, \alpha)$ is open and S_f is a random closed set. In particular, it implies $\pi = \pi_X$ measurable.

(c) $\Longrightarrow$ (m): Because for each $\xi \in \Xi$, $S_f(\xi)$ is an epigraph, for any set $D \subset X$,

$$\begin{aligned}\{\xi \in \Xi \mid \pi_D(\xi) < \beta\} &= \{\xi \in \Xi \mid S_f(\xi) \cap (D \times (-\infty, \beta)) \neq \emptyset\} \\ &= \{\xi \in \Xi \mid S_f(\xi) \cap (D \times (\alpha, \beta)) \neq \emptyset\} \ \ \forall \alpha < \beta.\end{aligned}$$

It is assumed that these sets belong to $\mathcal{S}$ when D is an open rational ball. Since every open set $O \subset X \times I\!R$ can be written as the countable union of sets of the type $O^\nu = I\!B^o(x^\nu, \rho^\nu) \times (\alpha^\nu, \beta^\nu)$ and $S_f^{-1}(O^\nu) \in \mathcal{S}$, it follows that

$$S_f^{-1}(O) = \bigcup_{\nu=1}^{\infty} S_f^{-1}(O^\nu) \in \mathcal{S}$$

which, by Proposition 3.3(b), implies that S_f is measurable. It's a random closed set since the lower semicontinuity of f implies that S_f is also closed-valued.

Now, let's assume that X is σ-compact or $\mathcal{S}$ is P-complete and let's show that under either one of these additional assumptions,

$$(\mathrm{m}) \Longrightarrow (\mathrm{d}) \Longrightarrow (\mathrm{e}) \Longrightarrow (\mathrm{f}) \Longrightarrow (\mathrm{m}).$$

Again, only the implications (m) $\Longrightarrow$ (d) and (f) $\Longrightarrow$ (m) need proof.

(m) $\Longrightarrow$ (d): It has already been established that if f is a random lsc function, $\pi = \pi_X$ is measurable. Given any closed set $C \subset X$, the function g with

$$g(\xi, x) = \begin{cases} f(\xi, x) & \text{if } x \in C, \\ \infty & \text{otherwise,} \end{cases}$$

is again a random lsc function since, in view of Proposition 3.3(c), when X is σ-compact or $\mathcal{S}$ is P-compete, for any closed set $D \subset X \times I\!R$,

$$S_g^{-1}(D) = \{\xi \in \Xi \,|\, S_f(\xi) \cap (C \times I\!R) \cap D \neq \emptyset\} = S_f^{-1}((C \times I\!R) \cap D) \in \mathcal{S}$$

and thus S_g is measurable and clearly closed-valued. Hence, $\xi \mapsto \inf g(\xi, \cdot) = \pi_C(\xi)$ is measurable as noted at the end of the proof that (m) $\Longrightarrow$ (a).

(f) $\Longrightarrow$ (m): It suffices to show that (f) $\Longrightarrow$ (a) since (a) $\Longrightarrow$ (m). Every open subset O of X can be written as the countable union of closed balls: with $x^\nu \in R$ and $\rho^\nu \in \mathbb{Q}_+$, one has

$$O = \bigcup_{\nu=1}^{\infty} I\!B(x^\nu, \rho^\nu), \qquad \pi_O = \inf\nolimits_\nu \pi_{I\!B(x^\nu, \rho^\nu)},$$

i.e., π_O can be obtained as the infimum of a countable collection of measurable function and consequently is measurable. ◻

Corollary 3.5 (countable scalarization)**.** *Let $f : \Xi \to$ lsc-fcns(X). For $x \in R$, a countable dense subset of X, and $\rho \in \mathbb{Q}_+$, define*

$$\pi_{x,\rho}(\xi) := \pi_{I\!B^o(x,\rho)}(\xi) = \inf\nolimits_{x \in I\!B^o(x,\rho)} f(\xi, x).$$

Then f is a random lsc function if and only if the countable collection of functions

$$\{\pi_{x,\rho} : \Xi \to \overline{I\!R} \,|\, x \in R, \rho \in \mathbb{Q}_+\}$$

are measurable.

When either X is σ-compact or $\mathcal{S}$ is P-complete, such a countable collection can also be obtained by replacing the open balls $\{I\!B^o(x, \rho) \,|\, x \in R, \rho \in \mathbb{Q}_+\}$ by their closed counterparts.

Proof. This is just a reformulation of parts (c) and (f) of the theorem. ◻

Thus, given a sequence of random lsc functions $\{f^\nu : \Xi \to$ lsc-fcns(X), $\nu \in I\!N\}$, one can always associate, by scalarization, a corresponding sequence of vector-valued random variables

$$\{\pi^\nu_{x,\rho},\ \nu \in I\!N \,|\, x \in R, \rho \in \mathbb{Q}_+\}.$$

As we demonstrate next, independence, stationarity and ergodicity properties of the sequence of the random lsc functions are inherited by these sequences of vectors generated by scalarization. To do so, we rely on Dynkin's π-λ theorem and some of its consequences which are briefly reviewed in Theorem 3.7, for details cf. [2].

Instead of restricting ourselves to scalar functions obtained via minimization over balls, we derive the results for minimization over arbitrary open sets.

Definition 3.6. *For a set Ω, $\mathcal{P}$ is a π-system (of subsets of Ω) if $\Omega \in \mathcal{P}$ and $\mathcal{P}$ is closed under intersections. $\mathcal{L}$ is a λ-system (of subsets of Ω) if it satisfies: $\Omega \in \mathcal{L}$, $B \setminus A \in \mathcal{L}$ for all A, $B \in \mathcal{L}$ such that $A \subset B$, and $A \in \mathcal{L}$ whenever $A = \bigcup_n A_n$ for a nested sequence $A_1 \subset A_2 \subset \cdots$ such that $A_n \in \mathcal{L}$.*

Theorem 3.7 (Dynkin's π-λ Theorem)**.** *Let $(\Omega, \mathcal{A}, \mu)$ be a probability space.*

(a) *If $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k$ are independent π-systems, then the (generated) σ-fields σ-$(\mathcal{P}_1)$, σ-$(\mathcal{P}_2)$,..., σ-$(\mathcal{P}_k)$ are also independent.*

(b) *If $\mathcal{P}$ is a π-system and $\mathcal{L}$ is a λ-system that contains $\mathcal{P}$, then σ-$(\mathcal{P}) \subset \mathcal{L}$.*

(c) *Let $\mathcal{P}$ be a π-system, P_1 and P_2 probability measures that agree on $\mathcal{P}$. Then P_1 and P_2 also agree on σ-$(\mathcal{P})$.*

For O^ν open subsets of X and $\alpha^\nu \in I\!\!R \cup \{\infty\}$, let

$$\mathcal{P} := \Big\{\mathcal{A} = \{f \in \text{lsc-fcns}(X) \,|\, \pi_{O^\nu} \le \alpha^\nu,\ \forall\, \nu \in I\!\!N\}\Big\}.$$

Observe that $\mathcal{P}$ is a generating class for the Effrös field $\mathcal{E}$ on lsc-fcns(X). Also define, the classes of product sets: for $k \in I\!\!N$,

$$\mathcal{P}^k = \mathcal{P} \times \mathcal{P} \times \cdots \times \mathcal{P} \qquad (k \text{ times})$$

and observe that

$$\sigma\text{-}(\mathcal{P}^k) = \mathcal{E}^k = \mathcal{E} \otimes \cdots \otimes \mathcal{E},$$

the σ-field generated by the product of k copies of $\mathcal{E}$.

Lemma 3.8. *For all $k \in I\!\!N$, $\mathcal{P}^k$ are π-systems on $(\text{lsc-fcns}(X))^k$.*

Proof. For any $k \in I\!\!N$ and $i \in \{1, \ldots, k\}$, one has

$$\mathcal{A}_i := \{f \in \text{lsc-fcns}(X) \,|\, \pi_{O_i^\nu} \le \infty \;\forall\, \nu \in I\!\!N\} = \text{lsc-fcns}(X),$$

and $\mathcal{A}_i \in \mathcal{P}$ whenever $\{O^\nu,\ \nu \in I\!\!N\}$ are open subsets of X. So, $\prod_{i=1}^k \mathcal{A}_i = (\text{lsc-fcns}(X))^k$, whereby $(\text{lsc-fcns}(X))^k \in \mathcal{P}^k$. Now for $k = 1$, arbitrary collections of open subsets of X and scalars in $I\!\!R \cup \{\infty\}$, $\{(O_1^\nu, \alpha_1^\nu),\ \nu \in I\!\!N\}$ and $\{(O_2^\nu, \alpha_2^\nu),\ \nu \in I\!\!N\}$, let

$$\mathcal{A}_1 := \{f \in \text{lsc-fcns}(X) \,|\, \pi_{O_1^\nu} \le \alpha_1^\nu,\ \forall\, \nu \in I\!\!N\} \in \mathcal{P},$$

$$\mathcal{A}_2 := \{f \in \text{lsc-fcns}(X) \,|\, \pi_{O_2^\nu} \le \alpha_2^\nu,\ \forall\, \nu \in I\!N\} \in \mathcal{P}.$$

Then

$$\mathcal{A}_1 \cap \mathcal{A}_2 = \{f \in \text{lsc-fcns}(X) \,|\, \pi_{O_i^\nu} \le \alpha_i^\nu,\ \forall\, \nu \in I\!N,\ i = 1, 2\} \in \mathcal{P},$$

whereby $\mathcal{P}$ is a π-system. For $k \in I\!N$, $i = 1, \dots, k$, let $\mathcal{A}_i^1$, $\mathcal{A}_i^2 \in \mathcal{P}$ and $\mathcal{A}^1 := \prod_{i=1}^k \mathcal{A}_i^1$, similarly $\mathcal{A}^2 := \prod_{i=1}^k \mathcal{A}_i^2$. Then using the fact that $\mathcal{P}$ is a π-system, we obtain

$$\mathcal{A}^1 \cap \mathcal{A}^2 = \prod_{i=1}^k (\mathcal{A}_i^1 \cap \mathcal{A}_i^2) \in \mathcal{P}^k$$

as claimed. □

Theorem 3.9. *Let (X, d) be a Polish space, $(\Xi, \mathcal{S}, P)$ a probability space, and f^1, f^2 random lsc functions defined on Ξ. Then f^1 and f^2 are identically distributed if and only if for all O^ν open subsets of X, $\alpha^\nu \in I\!R \cup \{\infty\}$, $\nu \in I\!N$:*

$$P\{\xi \in \Xi \,|\, \pi^1_{O^\nu}(\xi) \le \alpha^\nu,\ \forall\, \nu \in I\!N\} = P\{\xi \in \Xi \,|\, \pi^2_{O^\nu}(\xi) \le \alpha^\nu,\ \forall\, \nu \in I\!N\}.$$

Proof. Suppose f^1 and f^2 are identically distributed. Then for all $\mathcal{A} \in \mathcal{E}$,

$$P\{\xi \in \Xi \,|\, f^1(\xi, \cdot) \in \mathcal{A}\} = P\{\xi \in \Xi \,|\, f^2(\xi, \cdot) \in \mathcal{A}\},$$

hence this holds in particular for the sets

$$\mathcal{A} = \{f \in \text{lsc-fcns}(X) \,|\, \pi_{O^\nu} \le \alpha^\nu,\ \forall\, \nu \in I\!N\}.$$

For the reverse direction, suppose that for all $\nu \in I\!N$, O^ν open subsets of X, $\alpha^\nu \in I\!R \cup \{\infty\}$, one has

$$P\{\xi \in \Xi \,|\, \pi^1_{O^\nu}(\xi) \le \alpha^\nu,\ \forall\, \nu \in I\!N\} = P\{\xi \in \Xi \,|\, \pi^2_{O^\nu}(\xi) \le \alpha^\nu,\ \forall\, \nu \in I\!N\}.$$

For $i = 1$, 2, let μ_i be the measure induced by f^i on $\mathcal{E}$, i.e. for $\mathcal{A} \in \mathcal{E}$, $\mu_i(\mathcal{A}) = P\{\xi \in \Xi \,|\, f^i(\xi, \cdot) \in \mathcal{A}\}$. Then by supposition, $\mu_1 = \mu_2$ on $\mathcal{P}$. Since $\mathcal{P}$ generates the Effrös field, applying Theorem 3.7(c), yields $\mu_1 = \mu_2$ on $\mathcal{E}$. □

Theorem 3.10. *Let (X, d) be a Polish space, $(\Xi, \mathcal{S}, P)$ a probability space, and $\{f^\nu,\ \nu \in I\!N\}$ a sequence of random lsc functions defined on Ξ. Then, the sequence $\{f^\nu,\ \nu \in I\!N\}$ is independent if and only if for all $k \in I\!N$, indices $\ell_1, \dots, \ell_k$, scalars $\alpha_1^\nu, \dots, \alpha_k^\nu \in I\!R \cup \{\infty\}$ and $O_1^\nu, \dots, O_k^\nu$ open subsets of X, $\nu \in I\!N$,*

$$P\{\xi \in \Xi \,|\, \pi^{\ell_i}_{O_i^\nu}(\xi) \le \alpha_i^\nu,\ i = 1, \dots, k,\ \forall\, \nu \in I\!N\} = \prod_{i=1}^k P\{\xi \in \Xi \,|\, \pi^{\ell_i}_{O_i^\nu}(\xi) \le \alpha_i^\nu,\ \forall\, \nu \in I\!N\}.$$

In particular, for any open sets $O \subset X$, the sequence $\{\pi_O^\nu,\ \nu \in I\!N\}$ is independent whenever $\{f^\nu,\ \nu \in I\!N\}$ is independent.

Proof. Suppose $\{f^\nu,\ \nu \in I\!N\}$ is independent. Then for all $k, \nu \in I\!N$, indices $\ell_1, \ldots, \ell_k$, for all $\mathcal{A}_i \in \mathcal{E}$, $i = 1, \ldots, k$,

$$P\{\xi \in \Xi \,|\, f^{\ell_i}(\xi, \cdot) \in \mathcal{A}_i,\ i = 1, \ldots, k\} = \prod_{i=1}^{k} P\{\xi \in \Xi \,|\, f^{\ell_i}(\xi, \cdot) \in \mathcal{A}_i\},$$

hence this holds in particular for the sets

$$\mathcal{A}_i = \{f \in \text{lsc-fcns}(X) \,|\, \pi_{O_i^\nu} \le \alpha_i^\nu\ \forall\, \nu \in I\!N\}.$$

Suppose now that the asserted identity holds for all $k \in I\!N$, indices $\ell_1, \ldots, \ell_k$, scalars $\alpha_1^\nu, \ldots, \alpha_k^\nu$ in $I\!R \cup \{\infty\}$, and $O_1^\nu, \ldots, O_k^\nu$ open subsets of X, $\nu \in I\!N$. Fix $k \in I\!N$, $\ell_1, \ldots, \ell_k \in I\!N$ and let

$$\mathcal{P}_i := \text{ sets of the form } \{\xi \in \Xi \,|\, \pi_{O_i^\nu}^{\ell_i} \le \alpha_i^\nu,\ \forall\, \nu \in I\!N\}.$$

$\mathcal{P}_i$ is a π-system and σ-$(\mathcal{P}_i) = \sigma$-(f^{ℓ_i}). The indenpendence of the σ-(f^{ℓ_i}) follows from Theorem 3.7(a) which implies that the sequence $\{f^\nu,\ \nu \in I\!N\}$ is independent. □

Corollary 3.11. *Let (X, d) be a Polish space, $(\Xi, \mathcal{S}, P)$ a probability space and $\{f^\nu,\ \nu \in I\!N\}$ a sequence of random lsc functions defined on Ξ. Then, $\{f^\nu,\ \nu \in I\!N\}$ is iid if and only if the two following conditions are satisfied:*

(a) *for any pair f^l, f^k,*

$$P\{\xi \in \Xi \,|\, \pi_{O^\nu}^l(\xi) \le \alpha^\nu\ \forall\, \nu \in I\!N\} = P\{\xi \in \Xi \,|\, \pi_{O^\nu}^k(\xi) \le \alpha^\nu,\ \forall\, \nu \in I\!N\},$$

O^ν open subsets of X and $\alpha^\nu \in I\!R \cup \{\infty\}$, $\nu \in I\!N$;

(b) *for all $k \in I\!N$, indices $\ell_1, \ldots, \ell_k$, scalars $\alpha_1^\nu, \ldots, \alpha_k^\nu \in I\!R \cup \{\infty\}$ and $O_1^\nu, \ldots, O_k^\nu$ open subsets of X, $\nu \in I\!N$,*

$$P\{\xi \in \Xi \,|\, \pi_{O_i^\nu}^{\ell_i}(\xi) \le \alpha_i^\nu,\ i = 1, \ldots, k,\ \forall\, \nu \in I\!N\} = \prod_{i=1}^{k} P\{\xi \in \Xi \,|\, \pi_{O_i^\nu}^{\ell_i}(\xi) \le \alpha_i^\nu\ \forall\, \nu \in I\!N\}.$$

In particular, if $\{f^\nu,\ \nu \in I\!N\}$ is iid, then $\{\pi_O^\nu\}$ is iid for any O an open subset of X.

The next two theorems establish the stationarity and ergodicity of a sequence of random lsc functions through scalarization.

Theorem 3.12. *Let (X, d) be a Polish space, $(\Xi, \mathcal{S}, P)$ a probability space, and $\{f^\nu,\ \nu \in I\!N\}$ a sequence of random lsc functions defined on Ξ. Then $\{f^\nu,\ \nu \in I\!N\}$ is stationary if* *and only if for all $k,\ r \in \mathbb{N}$, indices $\ell_1, \ldots, \ell_k \in \mathbb{N}$, scalars $\alpha_1^\nu, \ldots, \alpha_k^\nu \in \mathbb{R} \cup \{\infty\}$ and $O_1^\nu, \ldots, O_k^\nu$ open subsets of X, $\nu \in \mathbb{N}$,*

$$P\{\xi \in \Xi \,|\, \pi^{\ell_i}_{O_i^\nu}(\xi) \leq \alpha_i^\nu,\ i = 1, \ldots, k,\ \forall\, \nu\} = P\{\xi \in \Xi \,|\, \pi^{\ell_i + r}_{O_i^\nu}(\xi) \leq \alpha_i^\nu,\ i = 1, \ldots, k,\ \forall\, \nu\}.$$

In particular, for any open sets $O \subset X$, the sequence $\{\pi_O^\nu,\ \nu \in \mathbb{N}\}$ is stationary whenever $\{f^\nu,\ \nu \in \mathbb{N}\}$ is stationary.

Proof. Suppose $\{f^\nu,\ \nu \in \mathbb{N}\}$ is stationary. Then for all $k,\ r \in \mathbb{N}$, indices $\ell_1, \ldots, \ell_k$, and for all $\mathcal{A}_i \in \mathcal{E}$, $i = 1, \ldots, k$,

$$P\{\xi \in \Xi \,|\, f^{\ell_i}(\xi, \cdot) \in \mathcal{A}_i,\ i = 1, \ldots, k\} = P\{\xi \in \Xi \,|\, f^{\ell_i + r}(\xi, \cdot) \in \mathcal{A}_i,\ i = 1 \ldots, k\},$$

hence this holds in particular for the sets

$$\mathcal{A}_i = \{f \in \text{lsc-fcns}(X) \,|\, \pi_{O_i^\nu} \leq \alpha_i^\nu\ \forall\, \nu \in \mathbb{N}\}.$$

Suppose now that for all $k, r \in \mathbb{N}$, indices $\ell_1, \ldots, \ell_k$, scalars $\alpha_1^\nu, \ldots, \alpha_k^\nu \in \mathbb{R} \cup \{\infty\}$, and $O_1^\nu, \ldots, O_k^\nu$ open subsets of X, $\nu \in \mathbb{N}$,

$$P\{\xi \in \Xi \,|\, \pi^{\ell_i}_{O_i^\nu}(\xi) \leq \alpha_i^\nu,\ i = 1, \ldots, k,\ \forall\, \nu\} = P\{\xi \in \Xi \,|\, \pi^{\ell_i + r}_{O_i^\nu}(\xi) \leq \alpha_i^\nu,\ i = 1, \ldots, k,\ \forall\, \nu\}.$$

For fixed $k, r, \ell_i \in \mathbb{N}$, $i = 1 \ldots, k$, let μ_1 be the measure induced on $\mathcal{E}^k$ by $(f^{\ell_1}, \ldots, f^{\ell_k})$. Let μ_2 be the measure induced on $\mathcal{E}^k$ by $(f^{\ell_1 + r}, \ldots, f^{\ell_k + r})$. By supposition, $\mu_1 = \mu_2$ on $\mathcal{P}^k$. Hence, by Theorem 3.7(c), $\mu_1 = \mu_2$ on $\mathcal{E}^k$. □

Theorem 3.13. *If $\{f \circ \varphi^\nu,\ \nu \in \mathbb{N}\}$ is an ergodic sequence of random lsc functions, then for all open $O \subset X$, $\{\pi_O \circ \varphi^\nu,\ \nu \in \mathbb{N}\}$ is an ergodic sequence of extended real-valued random variables.*

Proof. The shift operator, $\varphi : \text{lsc-fcns}(X)^\infty \to \text{lsc-fcns}(X)^\infty$ is ergodic, and π_O defined on $\text{lsc-fcns}(X)^\infty$ by $\pi_O(\zeta) := \inf_O \zeta_1$ is measurable. Hence the sequence, $\{\pi_O \circ \varphi^\nu\}$ is ergodic, and equivalent to the original sequence. □

4. Lower semicontinuity

We still work with (X, d) a Polish space and $(\Xi, \mathcal{S}, P)$ a probability space. We are going to show that given a random lsc function $f : \Xi \to \text{lsc-fcns}(X)$ and any σ-field $\mathcal{R} \subset \mathcal{S}$, there always is a version of the conditional expectation of f with respect to $\mathcal{R}$ that is lsc for all $\xi \in \Xi$.

Proposition 4.1. *Let R be a countable dense subset of X. A function $g : X \to \overline{\mathbb{R}}$ is lsc at $\bar{x}$ if and only if there exist $x^\nu \in R$ and $\rho^\nu \in \mathbb{Q}_+$ such that $x^\nu \to \bar{x}$, $\rho^\nu \searrow 0$, $d(x^\nu, \bar{x}) < \rho^\nu$ and*

$$\inf_{x \in B^o(x^\nu, \rho^\nu)} g(x) =: p_{x^\nu, \rho^\nu} \nearrow g(\bar{x}).$$

Proof. Simply observe that one can then choose a decreasing (sub)sequence

$$\{V^\nu := I\!B^o(x^\nu, \rho^\nu),\ x^\nu \in R, \rho^\nu \in Q_+\}$$

of neighborhoods of $\bar{x}$ such that $\bigcap_\nu V^\nu = \{\bar{x}\}$. And g is lsc at $\bar{x}$ if and only if $\inf_{V^\nu} g \nearrow g(\bar{x})$, as mentioned in §2. ◻

The existence of a lsc-version of the conditional expectation will be proved under a technical assumption that's also required to obtain the Ergodic Theorem in [5, §6].

Assumption 4.2. *A random lsc function $f : \Xi \to$ lsc-fcns(X) is locally inf-integrable if for every $x \in X$ there is a closed neighborhood V of x such that for the (scalar) function*

$$\xi \mapsto \pi_V(\xi) := \inf_{x' \in V} f(\xi, x') : \qquad E\{\pi_V\} > -\infty.$$

The assumption that f is a random lsc function already implies that given any closed set V, the function $\xi \mapsto \inf_V f(\xi, \cdot)$ is $\mathcal{S}$-measurable [7, Theorem 14.37].

Theorem 4.3. *Let $f : \Xi \to$ lsc-fcns(X) be a locally inf-integrable random lsc function and $\mathcal{R} \subset \mathcal{S}$ a σ-field. Then, there exists a version $E^{\mathcal{R}} f$ of the conditional expectation of f that is* lsc-fcns(X)*-valued, i.e., for all $\xi \in \Xi$, $E^{\mathcal{R}} f(\xi)$ is a lsc function.*

Proof. Let R be a countable dense subset of X and define

$$\{\pi_{x,\rho}(\xi) = \inf_{B^o(x,\rho)} f(\xi) \,|\, x \in R, \rho \in Q_+\},$$

a scalarization of f. Let $\pi^{\mathcal{R}}_{x,\rho} := E^{\mathcal{R}}\{\pi_{x,\rho}\}$ be a version of the conditional expectation of $\pi_{x,\rho}$ with respect to $\mathcal{R}$.

If $\pi_{x,\rho} \le \pi_{x',\rho'}$ P-a.s., then $\pi^{\mathcal{R}}_{x,\rho} \le \pi^{\mathcal{R}}_{x',\rho'}$ P-a.s. The inequality might fail on a set of measure 0. Since there are only a countable number of possible pairs (x, ρ, x', ρ'), the union of all such sets, i.e., on which the inequality doesn't hold, is of measure 0. So, let Ξ_0 be the subset of Ξ of P-measure 1 such that

$$\pi^{\mathcal{R}}_{x,\rho} \le \pi^{\mathcal{R}}_{x',\rho'} \text{ on } \Xi_0 \text{ whenever } \pi_{x,\rho} \le \pi_{x',\rho'} \ P\text{-a.s.}$$

Given $\bar{x} \in X$, choose $x^\nu \to \bar{x}$ with $x^\nu \in R$ and $\rho^\nu \searrow 0$ with $\rho^\nu \in Q_+$ such that $I\!B^o(x^\nu, \rho^\nu)$ is a neighborhood of both x^ν and $\bar{x}$, and $I\!B^o(x^{\nu+1}, \rho^{\nu+1}) \subset I\!B^o(x^\nu, \rho^\nu)$. By the lower semicontinuity of $f(\xi, \cdot)$ at $\bar{x}$, we know from Proposition 4.1 that $\pi_{x^\nu,\rho^\nu}(\xi) \nearrow f(\xi, \bar{x})$ for all $\xi \in \Xi$. In view of the above, for all $\xi \in \Xi_0$, the sequence $\{\pi^{\mathcal{R}}_{x^\nu,\rho^\nu}(\xi),\ \nu \in I\!N\}$ is monotone increasing. For $\xi \in \Xi$, define

$$f^{\mathcal{R}}(\xi, \bar{x}) := \lim_{\nu \to \infty} \pi^{\mathcal{R}}_{x^\nu,\rho^\nu}(\xi).$$

Let's first observe that the value assigned to $f^{\mathcal{R}}(\xi, \bar{x})$ is independent of the choice of the sequences $x^\nu \to \bar{x}$ and $\rho^\nu \searrow 0$. Indeed, let $\hat{x}^\nu \to \bar{x}$ and $\hat{\rho}^\nu \searrow 0$ be another pair of sequences satisfying the conditions: $x^\nu \in R$, $\rho^\nu \in \mathbb{Q}_+$, $\mathbb{B}^o(\hat{x}^\nu, \hat{\rho}^\nu)$ is a neighborhood of both $\hat{x}^\nu$ and $\bar{x}$ and $\mathbb{B}^o(\hat{x}^{\nu+1}, \rho^{\nu+1}) \subset \mathbb{B}^o(\hat{x}^\nu, \rho^\nu)$. Because both sequences of balls are decreasing neighborhoods of $\bar{x}$, for ν sufficiently large, $\mathbb{B}^o(x^\nu, \rho^\nu) \supset \mathbb{B}^o(\hat{x}^\mu, \hat{\rho}^\mu)$ for some $\mu \geq \nu$ and vice-versa. This implies that on Ξ_0, $\pi^{\mathcal{R}}_{x^\nu, \rho^\nu} \leq \pi^{\mathcal{R}}_{\hat{x}^\mu, \hat{\rho}^\mu}$ for some $\mu \geq \nu$ and vice-versa. Thus, both sequence must have the same limit.

Since $\pi_{x^\nu, \rho^\nu} \nearrow f(\cdot, \bar{x})$, the Monotone Convergence Theorem for conditional expectations, appealing here to local inf-integrability, implies that $f^{\mathcal{R}}$ is actually a version of the conditional expectation of f.

We show next that for all $\xi \in \Xi_0$, $f^{\mathcal{R}}(\xi, \cdot)$ is lsc. Consider a sequence $x^\nu \to \bar{x}$, $x^\nu \in R$, and pick a subsequence and $\rho^\nu \searrow 0$, $\rho^\nu \in \mathbb{Q}_+$, such that $\mathbb{B}^o(x^\nu, \rho^\nu)$ is a neighborhood of both x^ν and $\bar{x}$, and $\mathbb{B}^o(x^{\nu+1}, \rho^{\nu+1}) \subset \mathbb{B}^o(x^\nu, \rho^\nu)$. Then

$$f^{\mathcal{R}}(\cdot, x^\nu) \geq \pi^{\mathcal{R}}_{x^\nu, \rho^\nu} \text{ on } \Xi_0.$$

Taking liminf of both sides yields $\liminf_\nu f^{\mathcal{R}}(\cdot, x^\nu) \geq f^{\mathcal{R}}(\cdot, \bar{x})$. Since this holds for any such subsequence of $x^\nu \to \bar{x}$, it must hold for the sequence as well.

So far, we only considered sequences $x^\nu \to \bar{x}$ with $x^\nu \in R$. In the case of an arbitrary sequence $\bar{x}^\nu \to \bar{x}$ with $\bar{x}^\nu \in X$ note that it's always possible to find a sequence $x^\nu \to \bar{x}$ with $x^\nu \in R$, $d(\bar{x}^\nu, x^\nu)$ going sufficiently rapidly to 0 so that with an appropriate choice of $\rho^\nu \searrow 0$, for all ν, $\mathbb{B}^o(x^\nu, \rho^\nu)$ is not only a neighborhood of x^ν and $\bar{x}$ but also of $\bar{x}^\nu$. The same type of argument then yields $\liminf_\nu f^{\mathcal{R}}(\cdot, \bar{x}^\nu) \geq f^{\mathcal{R}}(\cdot, \bar{x})$. Thus, for all $\xi \in \Xi_0$, $f^{\mathcal{R}}(\xi, \cdot)$ is lsc.

Finally, for $\xi \in \Xi_0$, set $(E^{\mathcal{R}} f)(\xi) = f^{\mathcal{R}}(\xi, \cdot)$ and otherwise simply set $(E^{\mathcal{R}} f)(\xi) = \operatorname{cl} f^{\mathcal{R}}(\xi, \cdot)$ where $\operatorname{cl} f^{\mathcal{R}}(\xi, \cdot)$ is the lower semicontinuous closure of $f^{\mathcal{R}}(\xi, \cdot)$. The function $E^{\mathcal{R}} f : \Xi \to \text{lsc-fcns}(X)$ is then an lsc version of the conditional expectation of f. □

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