CONTINUITY PROPERTIES OF WALRAS EQUILIBRIUM POINTS *

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Abstract. We explore convergence notions for bivariate functions that yield convergence and stability results for their max/inf points. The results are then applied to obtain continuity results for Walras equilibrium points under perturbations of the utility functions of the agents.

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There is an abundant literature devoted to the study of the sensitivity/stability of the solutions, and the optimal value, of optimization problems to various perturbations. Epi-convergence theory [1, 4, 18] was developed to provide a comprehensive framework to deal with approximation and stability issues in the analysis of optimization problems.

On the other hand, the stability of market equilibrium prices has only received limited attention†. In this paper, we show that the stability of market equilibrium prices can be analyzed in a 'Variational Convergence' framework as well. In fact, two earlier papers [15, 12], analyzing economies with somewhat different properties, also derived continuity results for equilibrium prices by relying on epi-convergence tools; as far as we know, these articles are the only ones that deal directly with the stability of equilibrium prices in competitive economies; for comparison purposes a brief review of the argumenta used in [15, 12] can be found at the end of §4.

Our approach differs from [15, 12] in that we bring to the fore the notion of *converging economies.* To do this, we identify an economy with its *Walrasian* (function), a certain bivariate function that plays a role similar to that of the Lagrangian associated with an optimization problem, or the Hamiltonian associated with a Calculus of Variations problem. In particular, finding an equilibrium price for an economy can be translated as finding a max/inf point of the Walrasian. And, the convergence of economies can be formulated in terms of the (lopsided) convergence of their Walrasians; *lopsided convergence* was introduced in the mid-80's [2] but remained much on the back-shelf until now!

The Walrasian can be interpreted as the 'value function' of the so-called Walrasian auctioneer. There is also an intimate connection between the Walrasian and the variational inequality [8] associated with an economy.

This paper is aimed at introducing the framework and the associated methodology, and thus we won't work here with minimal assumptions; we leave this for a forthcoming paper [14] that also deals with a number of game models. So, here, we restrict ourselves to pure exchange economies and consider only perturbations of the utility functions of the agents. The assumptions we make about the economies are essentially those under which one usually obtains the existence of market equilibrium prices.

†There is however a substantial literature focused on generic stability properties with respect to initial endowment perturbations, see for example, Balasko [5], Mas-Colell [16] and more recently [6] and [7] and references therein. One could also view the study of the tatônnement process as providing some information about 'stability,' but in the absence of monotonicity, this route only brings information that is local in nature, one should refer to [21, 22], [13], [8] and related work such as [23], [11]; certain stability properties can also be found in [8] and [13].

1. A pure exchange economy

We are dealing with a pure exchange economy

$$\mathcal{E} = \{(u_a, e_a) \mid a \in \mathcal{A}\},$$

where

- $\mathcal{A}$: the set of agents;
- $e_a \in \mathbb{R}^n_+$: agent's $a \in \mathcal{A}$ endowment, a bundle of goods to be traded;
- $u_a : \mathbb{R}^n_+ \to \mathbb{R} \cup \{-\infty\} = (-\infty, \infty]$, agent's a utility function.

Trading takes place at a per-unit market price p_j for good j, $j = 1, \ldots, n$. The bundle of goods agent a could acquire is thus limited to those satisfying $\langle p, x\rangle \leq \langle p, e_a\rangle$, where $\langle \cdot, \cdot\rangle$ denotes the standard inner product. It's assumed that agents act as utility maximizers. Thus, given $p \in \mathbb{R}^n_+$, each agent $a \in \mathcal{A}$ will end up with the bundle of goods:

$$d_a(p) = \mathrm{argmax}_{x \in R^n} \{u_a(x) \mid \langle p, x\rangle \leq \langle p, e_a\rangle\};$$

our assumptions will imply that $d_a(p)$ is well-defined. Note that $d_a(p) = d_a(\alpha p)$ for any positive scalar α, i.e., multiplying the price vector p by a positive scalar doesn't affect the solution of the maximization problem. So, we may as well assume that $p \in \Sigma$ with $\Sigma = \{p \in \mathbb{R}^n_+ \mid \sum_{j=1}^n p_j = 1\}$, the *price simplex.*

This economy $\mathcal{E}$ is operational only if for each good, total supply exceeds total demand, i.e., if

$$\sum_{a \in \mathcal{A}} s_a(p) =: s(p) \geq 0 \quad \text{where} \quad s_a(p) := e_a - d_a(p).$$

The function $s : \mathbb{R}^n_+ \to \mathbb{R}^n$ is the *excess supply function.* A price vector $\bar{p} \in \Sigma$ so that $s(\bar{p}) \geq 0$ is called an *equilibrium price* (Walras, 1874)[‡] The existence of an equilibrium price isn't clear cut! Existence will be derived under classical assumptions as part of the ensuing development; it's a standard proof adapted to fit the present framework. We are interested in analyzing the sensitivity of equilibrium prices under perturbations of the agent's utilities.

We are not aiming, in this paper, at minimal assumptions. Our goal is to highlight the methodology; a projected article [14], that also deals with economies that include

[‡]In the Economics literature, one sometimes finds the equilibrium-condition $s(p) = 0$, but one can reasonably argue that the 'market' isn't concerned about some agents being left with some goods that have no takers.

production and also analyzes Nash equilibrium points of non-cooperative games, provides a more comprehensive, and more technical, development.

Utility functions. For each agent $a \in \mathcal{A}$, $u_a : \mathbb{R}^n_+ \to \mathbb{R} \cup \{-\infty\}$ is usc, strictly concave on $\operatorname{dom} u_a \subset \mathbb{R}^n_+$ that is assumed to be closed and have nonempty interior, i.e., $\operatorname{int} \operatorname{dom} u_a \neq \emptyset$. These are benign assumptions. Slightly more questionable is the assumption that u_a is sup-compact, i.e., that its (super-)level sets

$$\operatorname{lev}_\alpha u_a = \{x \mid u_a(x) \geq \alpha\}$$

are bounded in addition to being closed, as follows from the upper semicontinuity assumption. Here is what this assumption imply about our economic agents: There is an optimal bundle of goods that each agent will strive to acquire via trading. Further acquisitions of goods beyond this optimal bundle would not only be useless but would actually result in decreased utility due, for example, to the need to store or dump the surplus. From a technical viewpoint, the assumption that u_a is sup-compact, will guarantee the existence of a solution to the agent's probem without having to impose directly a 'boundedness' restriction on the range of goods that the agent is capable of acquiring.

The (effective) domain of u_a, $\operatorname{dom} u_a$, represents the set of *survival bundles* for agent a, i.e., consumption vectors that would enable agent a to 'survive'. So, for a given price $p \in \Sigma$, the agent's maximization problem:

$$\max u_a(x) \ \text{ so that } \ \langle p, x - e_a \rangle \leq 0,$$

either has a unique optimal solution $d_a(p)$ or the problem is infeasible. Let's denote the set of feasible trades by

$$R_a(p) := \operatorname{dom} u_a \cap \{x \mid \langle p, x - e_a \rangle \leq 0\}.$$

When $R_a(p) = \emptyset$, the bundle of goods e_a can't be traded for a bundle of goods that would ensure the survival of agent a.

Economy. We assume that there is a finite number $|\mathcal{A}|$ of agents. Thus, our *economy* $\mathcal{E}$ consists of finite number of pairs: $\{(u_a, e_a), a \in \mathcal{A}\}$. We shall analyze the sensitivity of equilibrium market prices to perturbations of the agents utility functions.

Consistency assumption. In keeping with the 'standard' literature on Walras equilibrium models [9], we assume that for all $a \in \mathcal{A}$, $e_a \in \operatorname{int} \operatorname{dom} u_a$[§]. This *consistency*

[§] An implication of this assumption is that even before any trading takes place all the agents possess a bundle of goods that will guarantee their survival. From an economic viewpoint this is a questionable assumption to which we plan to return in further work.

assumption has important technical implications: It's immediate that for all $p \in \Sigma$ and for all agents $a \in \mathcal{A}$, $R_a(p)$ is a nonempty closed convex set. But much more can be asserted.

A set-valued mapping $R\colon \Sigma \rightrightarrows I\!\!R^n_+$ is *continuous relative to* Σ if for all $p \to \bar p \in \Sigma$:

$$R(p) \to R(\bar p) \quad \text{which means} \quad \limsup_{p\to\bar p} R(p) \subset R(\bar p) \subset \liminf_{p\to\bar p} R(p);$$

where $\limsup_{p\to\bar p} S(p)$ and $\liminf_{p\to\bar p} R(p)$ denote the (standard) outer and inner limits of families of sets, see [18, Chapters 4 & 5] for more about set convergence and set-valued mappings.

Proposition 1. *For all $a \in \mathcal{A}$, the mapping $R_a\colon \Sigma \rightrightarrows I\!\!R^n_+$ is closed-, convex-valued and such that for all $p \in \Sigma$, $\operatorname{int} R_a(p) \neq \emptyset$. Moreover, it's continuous relative to Σ.*

Proof. That R_a is closed-, convex-valued is immediate from the the definition. Since $e_a \in \operatorname{int}(\operatorname{dom} u_a)$, it follows immediately that $\operatorname{int}(\operatorname{dom} u_a) \cap \{x \mid \langle p, x\rangle < \langle p, e_a\rangle\} \neq \emptyset$ for all $p \in \Sigma$, which implies that $\operatorname{int} R_a(p)$ is nonempty.

To see that $\limsup_{p\to\bar p} R_a(p) \subset R_a(\bar p)$, or in other words, that the mapping is osc (outer semicontinuous), simply observe that if $p^\nu \to \bar p$ in Σ and $x^\nu \to \bar x$ then

$$\langle p^\nu, x^\nu - e_a\rangle \;\to\; \langle \bar p, \bar x - e_a\rangle,$$

and consequently, if $x^\nu \in R_a(p^\nu)$, then $\bar x \in R_a(\bar p)$; recall that $\operatorname{dom} u_a$ is closed.

To verify that $\liminf_{p\to\bar p} R_a(p) \supset R_a(\bar p)$, or in other words, that the mapping is isc (inner semicontinuous), we appeal to the criterion provided by [18, Theorem 5.9(a)]. Indeed, if $\bar x \in \operatorname{int} R_a(\bar p)$, it implies that for some $\eta > 0$, $I\!\!B(\bar x, \eta) \subset R_a(\bar p)$ and for all $x \in I\!\!B(\bar x, \eta)$, $\langle \bar p, \bar x - e_a\rangle < 0$. Thus, one can certainly find a neighborhood W of $(\bar p, \bar x)$ such that $W \cap (\Sigma \times I\!\!R^n) \subset \operatorname{gph} R_a$. Since R_a is closed-, convex-valued, [18, Theorem 5.9(a)] implies that R_a is isc at every point $\bar p$ of Σ. This completes the proof since we already established that R_a is outer semicontinuous on Σ. $\square$

Perturbations and convergent economies. We consider economies

$$\mathcal{E}^\nu = \{(u^\nu_a, e_a), a \in \mathcal{A}\} \qquad u^\nu_a : I\!\!R^n \to I\!\!R \cup \{-\infty\}, \quad \nu \in I\!\!N,$$

with the utility functions u^ν_a converging *continuously* to u_a: for any $\{x^\nu \to x, \nu \in I\!\!N\}$ in $\operatorname{dom} u_a$, $u^\nu_a(x^\nu) \to u_a(x)$. It's also assumed that for all ν, $\operatorname{dom} u^\nu_a = \operatorname{dom} u_a$ and that on $\operatorname{dom} u_a$, the function u^ν_a possess the same properties as u_a. One of the application we have in mind is when the utility functions are perturbed linearly, i.e., $u^\nu_a = u_a + \langle w^\nu, \cdot\rangle$ with $w^\nu \to 0$.

Given the assumptions we have made about u_a, viz. concavity and $\operatorname{int}\operatorname{dom} u_a \neq \emptyset$, continuous convergence doesn't amount to much more than pointwise convergence, the only additional requirement imposed by continuous convergence has to do with what happens on the boundary of $\operatorname{dom} u_a$ [18, Theorem 7.17, Theorem 7.14]. Continuous convergence demands that the convergence on the boundary be 'consistent' with the (pointwise) convergence on $\operatorname{int}\operatorname{dom} u_a$.

2. The excess supply function

We show next that for economies $\mathcal{E}^\nu$ converging to $\mathcal{E}$, the corresponding excess supply function converge continuously. We begin by appealing to the theory of *hypo-convergence.*

Once the market price is fixed, each agent is faced with a maximization problem. And, any optimization problem, involving constraints or not, can always be expressed as the maximization of an extended real-valued function. In the case of a constrained-maximization problem, simply redefine the objective as taking on the value $-\infty$ outside the feasible region (the set determined by the constraints). In this framework, the canonical problem can be formulated as one of maximizing a single, extended real-valued, upper semicontinuous (usc) function $f : I\!R^n \to \overline{I\!R}$; the upper semicontinuity coming from having the (given) objective usc and the feasible region closed. Approximation issues can consequently be studied in terms of the convergence of usc functions. This leads us to the notion of *hypo-convergence* which plays a key role in 'Variational Analysis' [4, 18]; the theory is usually presented in a minimization framework and one deals then with lower semicontinuous (lsc) function and convergence is formulated in terms of *epi*-convergence.

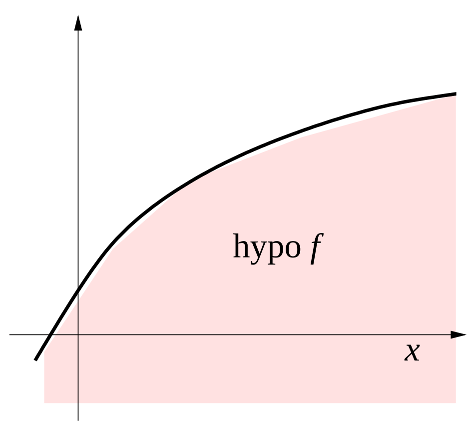

Figure 1: The hypograph of a concave function f

Definition 2. *A sequence of functions* $\{f^\nu : \mathbb{R}^n \to \overline{\mathbb{R}},\ \nu \in \mathbb{N}\}$ *hypo-converges to* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, *written* $f^\nu \xrightarrow{h} f$, *if for all* $x \in \mathbb{R}^n$:

(a) $x^\nu \to x$ *implies* $\limsup_\nu f^\nu(x^\nu) \le f(x)$,

(b) $\exists\ x^\nu \to x$ *such that* $\liminf_\nu f^\nu(x^\nu) \ge f(x)$.

The sequence is said to epi-converge when $-f^\nu \xrightarrow{h} -f$.

This convergence notion is equivalent to the set-convergence (in the Painlevé-Kuratowski sense) of the hypographs of the f^ν to the hypographs of f, the *hypograph* of a function is the set of all points that lie on or below the graph of the function, see Figure 1. This property motivates the use of the term 'hypo-convergence'. The implications of hypo-convergence that are of immediate concern are summarized in the next lemma.

Lemma 3 [18, Theorem 7.31]. *Suppose* $\{f, f^\nu : \mathbb{R}^n \to \overline{\mathbb{R}},\ \nu \in \mathbb{N}\}$ *are such that* $f^\nu \xrightarrow{h} f$, *then*

$$\liminf_{\nu\to\infty} (\sup f^\nu) \ \ge\ \sup f.$$

Moreover, if there is a subsequence $\{\nu_k\}_{k\in\mathbb{N}}$, *such that for all* k, $x^k \in \operatorname{argmax} f^{\nu_k}$ *and* $x^k \to \bar{x}$, *then* $\bar{x} \in \operatorname{argmax} f$ *and also* $\sup f^{\nu_k} \to \sup f$.

In particular, if $\operatorname{argmax} f$ *is a singleton, then every convergent subsequence of the* $\{x^\nu \in argmax f^\nu,\ \nu \in \mathbb{N}\}$ *converges to* $\operatorname{argmax} f$.

Let's now return to our agent's problem and consider what happens under perturbations of the market prices.

Proposition 4. *For all agents in an economy* $\mathcal{E}$ *that satisfies the consistency assumption, the demand function* $p \mapsto d_a(p) : \Sigma \to \mathbb{R}^n$ *is continuous.*

This is an immediate corollary of the next proposition that considers the effect on the demand functions of perburbing both the utility functions and the prices p.

Proposition 5. *Let* $\mathcal{E}$ *satisfy the consistency assumption:* $e_a \in \operatorname{int}(\operatorname{dom} u_a)$ *for all* $a \in \mathcal{A}$. *For* $p, p^\nu \in \Sigma$, *define*

$$d_a(p) = \operatorname{argmax}\{u_a(x) \,|\, x \in R_a(p)\},$$
$$d_a^\nu(p^\nu) = \operatorname{argmax}\{u_a^\nu(x) \,|\, x \in R_a(p^\nu)\}.$$

Assume that $\{u_a^\nu,\ \nu \in \mathbb{N}\}$ *is a sequence of utility functions such that* $u_a^\nu \xrightarrow{c} u_a$. *Then* $d_a^\nu(p^\nu) \to d_a(p)$ *for any sequence* $p^\nu \to p$ *in* Σ; *this means that* $d_a^\nu \xrightarrow{c} d_a$ *relative to* Σ.

Proof. With $p^\nu \to p$ in Σ, define

$$v(x) = \begin{cases} u_a(x) & \text{if } \langle p, x\rangle \le \langle p, e_a\rangle, \\ -\infty & \text{otherwise }, \end{cases}$$

and, for $\nu = 1, \ldots,$

$$v^\nu(x) = \begin{cases} u_a^\nu(x) & \text{if } \langle p^\nu, x\rangle \le \langle p^\nu, e_a\rangle, \\ -\infty & \text{otherwise }, \end{cases}$$

The properties of the utility functions mentioned in §1 imply that the argmax in the definition of $d_a(p)$ and $d_a^\nu(p^\nu)$ are singletons.

We have to verify: for all x,

- (a) for all $x^\nu \to x$, $\limsup_\nu v^\nu(x^\nu) \le v(x)$,
- (b) there exists $x^\nu \to x$ such that $\liminf_\nu v^\nu(x^\nu) \ge v(x)$.

Beginning with (a): This is evident when $\limsup_\nu v^\nu(x^\nu) = -\infty$. Otherwise for ν sufficiently large, $x^\nu \in R_a(p^\nu)$ and consequently $x \in R_a(p)$ since $p^\nu \to p$ and then, the inequality simply follows from $u_a^\nu(x^\nu) \to u_a(x)$ by the continuous convergence assumption.

Turning to (b), with $x^\nu \equiv x$, the inequality $\liminf v^\nu(x^\nu) \ge v(x)$ is trivially satisfied if $x \notin R_a(p)$. If $x \in \operatorname{int} R_a(p)$, then for ν sufficiently large, $x^\nu \in \operatorname{int} R_a(p^\nu)$ and consequently $v^\nu(x^\nu) \to v(x)$, recall that $x^\nu \equiv x$. Finally, if $x \in \operatorname{bdry} R_a(p)$, from the properties of R_a and appealing again to Proposition 1 enables us to find a sequence $x^\nu \to x$ with $x^\nu \in \operatorname{int} R_a(p^\nu)$ that yields the desired inequality.

Thus, $v^\nu \xrightarrow{h} v$ and whenever $R_a(p) \neq \emptyset$, $p^\nu \to p$,

$$d_a^\nu(p^\nu) = \operatorname{argmax} v^\nu \to \operatorname{argmax} v = d_a(p). \qquad \square$$

Corollary 6. *Let $\mathcal{E}$ be an economy satisfying our consistency assumption. Suppose that for all $a \in \mathcal{A}$, the sequence of $u_a^\nu \xrightarrow{c} u_a$ and $p^\nu \to p$ with $p^\nu \in \Sigma$. Then*

$$\sum_{a\in\mathcal{A}} e_a - \sum_{a\in\mathcal{A}} d_a^\nu(p^\nu) := s^\nu(p^\nu) \;\to\; s(p) := \sum_{a\in\mathcal{A}} e_a - \sum_{a\in\mathcal{A}} d_a(p),$$

i.e., $s^\nu \xrightarrow{c} s$ on Σ; the excess supply functions converge continuously relative to Σ.

Proof. Follows directly from the preceding proposition. $\square$

3. Lopsided convergence.

One can translate the problem of finding an equilibrium market price into one of finding max/inf point of a certain bivariate function, see [3, Chapter 6]. In that framework, sensitivity analysis of market prices becomes one of the sensitivity of the max/inf points to perturbation of this bivariate function.

Lopsided convergence for bivariate functions [2] is aimed at the convergence of max/inf points, or min/sup points, but not both. Consequently, the definition is necessarily one-sided. Here we are interested in max/inf points, so the definition of lopsided is formulated in that context.

For $F : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}} = [-\infty, \infty]$, one refers to $\bar{x}$ as a *max/inf point* if

$$\bar{x} \in \operatorname*{argmax}_{x \in \mathbb{R}^n} \big[\inf_{y \in \mathbb{R}^m} F(x,y) \big];$$

it's a *min/sup point* if

$$\bar{x} \in \operatorname*{argmin}_{x \in \mathbb{R}^n} \big[\sup_{y \in \mathbb{R}^m} F(x,y) \big].$$

Definition 7. *A sequence of bivariate functions* $\{F^\nu : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}, \nu \in \mathbb{N}\}$ *converges lopsided to a function* $F : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$ *if for all* $(x,y) \in \mathbb{R}^n \times \mathbb{R}^m$*:*

(a) *for any* $x^\nu \to x$*, there exists* $y^\nu \to y$ *such that* $\limsup_\nu F^\nu(x^\nu, y^\nu) \le F(x,y)$*,*

(b) *there exists* $x^\nu \to x$ *such that for all* $y^\nu \to y$*,* $\liminf_\nu F^\nu(x^\nu, y^\nu) \ge F(x,y)$*.*

In the proof of our next result, we rely on a further refinement of Lemma 3.

Lemma 8 [18, Theorem 7.31]**.** *Suppose* $h^\nu \xrightarrow{h} h$ *with* $\inf f \in \mathbb{R}$*. If there is a compact set* $B \subset \mathbb{R}^n$ *and an index* ν_B *such that*

$$\forall\, \nu \ge \nu_B : \quad \inf_B h^\nu = \inf h^\nu,$$

then $\inf h^\nu \to \inf h$*.*

Theorem 9. *Suppose the sequence of bivariate functions* $\{F^\nu : C \times D \to \mathbb{R}, \nu \in \mathbb{N}\}$ *converges lopsided* $F : C \times D \to \mathbb{R}$ *with* $C \subset \mathbb{R}^n$ *and* $D \subset \mathbb{R}^m$ *compact, and such that* $\inf_y F(\cdot, y) < \infty$*. Then, the functions* $\inf_y F^\nu(\cdot, y)$ *hypo-converge to* $\inf_y F(\cdot, y)$*.*

Proof. Let

$$g^\nu(x) = \inf\nolimits_{y \in \mathbb{R}^m} F^\nu(x,y), \quad g(x) = \inf\nolimits_{y \in \mathbb{R}^m} F(x,y).$$

We have to show that $g^\nu \xrightarrow{h} g$. Let's begin with the condition: $\limsup_\nu g^\nu(x^\nu) \le g(x)$ for any arbitrary sequence $x^\nu \to x$. We argue separately the cases $g(x) \in \mathbb{R}$ and $g(x) = -\infty$; by assumption $g < \infty$. If $g(x) \in \mathbb{R}$, for $\varepsilon > 0$, let $y_\varepsilon \in \varepsilon\text{-}\operatorname{argmin} F(x, \cdot)$ and let $y^\nu \to y_\varepsilon$ be such that $\limsup_\nu F^\nu(x^\nu, y^\nu) \le F(x, y_\varepsilon)$. From condition (a) in the Definition 7 of lopsided convergence, one has

$$\limsup_\nu g^\nu(x^\nu) \le \limsup_\nu F^\nu(x^\nu, y^\nu) \le F(x, y_\varepsilon) \le g(x) + \varepsilon.$$

Since, this holds for arbitrary $\varepsilon > 0$, it follows that $\limsup_\nu g^\nu(x^\nu) \leq g(x)$. When $g(x) = -\infty$, for any $\kappa < 0$ there is a y_κ such that $F(x, y_\kappa) < \kappa$. Again by 7(a), can find $y^\nu \to y_\kappa$ such that

$$\limsup_\nu g^\nu(x^\nu) \leq \limsup_\nu F^\nu(x^\nu, y^\nu) \leq F(x, y_\kappa) \leq \kappa.$$

Since this holds for all $\kappa < 0$, it follows that $\limsup_\nu g^\nu(x^\nu) = -\infty$.

Let's now turn to the second condition for hypo-convergence, namely, there exists $x^\nu \to x$ such that $\liminf_\nu g^\nu(x^\nu) \geq g(x)$. The inequality is trivially satisfied if $g(x) = -\infty$, so let's assume henceforth that $g(x) \in I\!R$; the case $g(x) = \infty$ has been excluded by assumption. Let $x^\nu \to x$ be a sequence predicated by condition (b) of lopsided convergence. Now, from (b) and (a), it follows that the functions $\{F^\nu(x^\nu, \cdot) : I\!R^m \to \overline{I\!R},\ \nu \in I\!N\}$ epi-converge to $F(x, \cdot)$. There remains only to appeal to Lemma 8 and the definitions of g^ν, g to conclude that $g^\nu(x^\nu) \to g(x)$. ◻

Corollary 10. *Let $\{F^\nu : C \times D \to I\!R\ \nu \in I\!N\}$ be a sequence of bivariate functions that converge lopsided to a function F with $C \subset I\!R^n$ and $D \subset I\!R^m$ compact, and such that $\inf_y F(\cdot, y) < \infty$. If for all ν, x^ν is a* max/inf point of F^ν and $\bar{x}$ is any cluster point of the sequence $\{x^\nu,\ \nu \in I\!N\}$, then $\bar{x}$ is a max/inf point of the limit function F. Moreover, with $\{x^\nu, \nu \in N\}$, the subsequence converging to $\bar{x}$,

$$\lim_{\nu \underset{N}{\to} \infty} \Big[\inf_{y \in \boldsymbol{R}^m} F^\nu(x^\nu, y) \Big] = \inf \big[\inf_{y \in \boldsymbol{R}^m} F(\bar{x}, y) \big],$$

i.e., there is also convergence of the values of the max/inf points.

Proof. Theorem 9 tells us that with

$$g^\nu(x) = \inf\nolimits_{y \in \boldsymbol{R}^m} F^\nu(x, y), \quad g(x) = \inf\nolimits_{y \in \boldsymbol{R}^m} F(x, y),$$

the functions g^ν hypo-converge to g. Max/inf points for F^ν and F are then maximizers of the corresponding functions g^ν and g. The assertions now follow immediately from Lemma 3. ◻

The class of usc functions is closed under hypo-converge [18, Theorem 7.4], and so is the class of concave usc functions [18, Theorem 7.17]. The class of Ky Fan functions is closed under lopsided convergence.

Definition 11. *A bivariate function $F : I\!R^n \times I\!R^n \to \overline{I\!R}$ is a Ky Fan function if*

(a) *for all $y \in I\!R^n$, $x \mapsto F(x, y)$ is usc;*
(b) *for all $x \in I\!R^n$, $y \mapsto F(x, y)$ is convex;*
(c) *F is finite-valued on $K \times K$ with $K \subset I\!R^n$ compact (and necessarily convex).*

For functions of this type, Ky Fan derived the following important existence result:

Theorem 12 [10], [3, Theorem 6.3.5]. *Suppose $F : I\!\!R^n \times I\!\!R^n \to \overline{I\!\!R}$ is a Ky Fan function with $K \times K$ the compact set on which F is finite-valued. Then, the set of* max/inf points is a nonempty subset of K. Moreover, if $F(x,x) \geq 0$ on $K \times K$ then for every max/inf point $\bar{x}$, one has $\inf_y F(\bar{x}, y) \geq 0$.

For more about equilibrium points and Ky Fan inequality, refer to [19] and [20].

Theorem 13. *Let $\{F^\nu : I\!\!R^n \times I\!\!R^n \to \overline{I\!\!R},\ \nu \in I\!\!N\}$ be a sequence of Ky Fan functions with $K \times K$, the compact subset of $I\!\!R^n \times I\!\!R^n$, on which they are finite-valued. Suppose these functions converge lopsided to a function F. Then, F is a Ky Fan function, i.e., the class of Ky Fan functions is closed under lopsided convergence.*

For each ν, the set of max/inf points of F^ν is nonempty.

For $\nu \in I\!\!N$, let x^ν be a max/inf point of F^ν and let $\bar{x}$ be a cluster point of the sequence $\{x^\nu,\ \nu \in I\!\!N\}$. Every such cluster point is a max/inf point of F. Finally, if for all ν, $F^\nu(y,y) \geq 0$, then for $F(\bar{x}, \cdot) \geq 0$.

Proof. In view of Theorem 12, Corollary 10 and because $K \subset I\!\!R^n$ is compact, the only assertion that really needs proof is: F is a Ky Fan function that is finite-valued on $K \times K$, and that follows immediately from lopsided convergence. □

4. Walras equilibrium: sensitivity

An equilibrium price $\bar{p}$ is a solution of the inclusion

$$S(p) \ni 0 \quad \text{where} \quad S : \Sigma \rightrightarrows I\!\!R^n \quad \text{with} \quad S(p) = s(p) - I\!\!R^n_+.$$

Our analysis follows a pattern first laid out by Aubin and Ekeland in [3, Chapter 6]. Let's introduced the function $W : \Sigma \times \Sigma \to I\!\!R$ where

$$W(p,q) = \sup\{\langle q, s\rangle \,\big|\, s \in S(p)\}.$$

We refer to this function as the *Walrasian* associated with the economy $\mathcal{E}$[¶]. This bivariate function W has the following properties:

- (a) $\forall\, q \in \Sigma : \quad W(\cdot, q)$ is usc,

[¶]This function plays a role similar to that of the Lagrangian function associated with an optimization problem. In particular, the convergence of economies can be interpreted, as done here, in terms of the convergence of their Walrasian functions just like the convergence of optimization problems can be characterized in terms of the convergence of their Lagrangian functions. The designation *Walrasian* comes in part from the fact that it is the 'value' function of the so-called *Walrasian auctioneer*, but more significantly, it encapsulates all the essential mathematical properties of the economic model being considered, and in view of this, no other name seemed appropriate.

- (b) $\forall\, p \in \Sigma: \quad W(p, \cdot)$ is convex,
- (c) $\forall\, q \in \Sigma: \quad W(q, q) \geq 0$.

To see that (a) holds, simply observe that $\sup_{s\in S(p)}\langle q, s\rangle = \langle q, s(p)\rangle$ and $p \mapsto s(p)$ is continuous, cf. Proposition 4. For (b): $W(p, \cdot)$ is the supremum of a collection of linear functions. Finally, for (c), given any price $q \in \Sigma$, for each agent $a \in \mathcal{A}$, one has $\langle q, d_a(q)\rangle \leq \langle q, e_a\rangle$ and hence, $\langle q, s(q)\rangle \geq 0$.

Because Σ is compact, in view of properties (a) and (b), the Walrasian W is a Ky Fan function. And, for Ky Fan functions, the existence of max/inf points is guaranteed by Ky Fan 's Inequality:

Theorem 14 (Ky Fan's Inequality). *The Ky Fan function W has a* max/inf point $\bar{p} \in \Sigma$ such that

$$0 \leq \inf_{q\in\Sigma} W(\bar{p}, q) = \sup_{p\in\Sigma}\inf_{q\in\Sigma} W(p, q).$$

Moreover, this max/inf point $\bar{p}$ is an equilibrium price, i.e., $S(\bar{p}) \ni 0$.

Proof. The first assertion follows from [3, Lemma 6.3] and the subsequent Remark.

To see that $\bar{p}$ is also an equilibrium price it really suffices to observe that $s(\bar{p}) \geq 0$ which certainly implies that $S(\bar{p}) \ni 0$. Indeed, since $W(\bar{p}, q) = \langle q, s(\bar{p})\rangle \geq 0$ for all $q \in \Sigma$, in particular $\langle e^j, s(\bar{p})\rangle \geq 0$ where $e^j = (0, \ldots, 0, 1, 0, \ldots, 0)$, i.e., e^j is the unit vector whose jth entry is 1. Thus, $s_j(\bar{p}) \geq 0$ for $j = 1, \ldots, n$. $\square$

We are now all set to consider a sequence

$$\Big\{\mathcal{E}^\nu = \big\{(u_a^\nu, e_a), a \in \mathcal{A}\big\},\ \nu \in I\!N\Big\}$$

of economies converging to an economy $\mathcal{E} = \big\{(u_a, e_a), a \in \mathcal{A}\big\}$. Market equilibrium prices have remarkable continuity properties:

15 Theorem. *Consider the sequence of economies $\mathcal{E}^\nu = \big\{(u_a^\nu, e_a), a \in \mathcal{A}\big\}_{\nu\in N}$ and $\mathcal{E} = \big\{(u_a, e_a), a \in \mathcal{A}\big\}$ that satisfies the consistency assumption. Suppose that for all ν, $\operatorname{dom} u_a^\nu = \operatorname{dom} u_a$ for all $a \in \mathcal{A}$ which implies that for all ν, the economy $\mathcal{E}^\nu$ also satisfies the consistency assumption. Suppose that for each agent $a \in \mathcal{A}$, the utility functions $u_a^\nu \xrightarrow{c} u_a$. Then, each economy $\mathcal{E}^\nu$ and $\mathcal{E}$ has at least one equilibrium price in Σ, denoted $\bar{p}^\nu$ or $\bar{p}$. The set of cluster points of the sequence $\{\bar{p}^\nu\}_{\nu\in N}$ is never empty, and any such cluster point, say $\bar{p}$, is a market equilibrium price for $\mathcal{E}$.*

Proof. To each one of these economies, associate its Walrasian function

$$W(p, q) = \sup\{\langle q, s\rangle \,\big|\, s \in S(p)\} \quad \text{with} \quad S(p) = s(p) - I\!R^n_+,$$
$$W^\nu(p, q) = \sup\{\langle q, s\rangle \,\big|\, s \in S^\nu(p)\} \quad \text{with} \quad S^\nu(p) = s^\nu(p) - I\!R^n_+,$$

where s^ν is the excess supply function associated with $\mathcal{E}^\nu$ and s is the excess supply function associated with $\mathcal{E}$. We know that these Walrasian functions are Ky Fan functions defined on the product $\Sigma \times \Sigma$ of (nonempty) compact convex sets. Equilibrium prices correspond to max/inf points for these Walrasian functions. And this means that the assertions of this theorem follow from Corollary 10 and Theorem 13 once we show that the Walrasian functions W^ν lopsided converge to W. We need to verify:

(a) $\forall\,(p,q) \in \Sigma \times \Sigma$, $\forall, p^\nu \to p$, $\exists\, q^\nu \to q$ such that $\limsup_{\nu\to\infty} W^\nu(p^\nu, q^\nu) \le W(p,q)$;

(b) $\forall\, p \in \Sigma$, $\exists\, p^\nu \to p$ such that $\forall\, q^\nu \to q$ in Σ, $\liminf_{\nu\to\infty} W^\nu(p^\nu, q^\nu) \ge W(p,q)$.

For (a), since

$$W^\nu(p^\nu, q^\nu) = \sup\left[\,\langle q^\nu, s\rangle \,\middle|\, s \in S^\nu(p^\nu)\,\right] = \langle q^\nu, s^\nu(p^\nu)\rangle$$

and from Corollary 6, $s^\nu \xrightarrow{c} s$, it follows that for any sequence $q^\nu \to q$ in Σ,

$$\langle q^\nu, s^\nu(p^\nu)\rangle \to \langle q, s(p)\rangle = W(p,q).$$

For (b), let $p^\nu \equiv p$. For any $q^\nu \to q$ in Σ, one has

$$W(p^\nu, q^\nu) = \langle q^\nu, s^\nu(p)\rangle \;\to\; \langle q, s(p)\rangle = W(p,q)$$

since $s^\nu \xrightarrow{c} s$ implies $s^\nu \xrightarrow{p} s$. $\square$

Instead of this proof, one could also argue as follows: Under the same hypotheses, the existence of equilibrium prices for each one of the economies $\{\mathcal{E}^\nu, \nu \in I\!N\}$ and $\mathcal{E}$ is guaranteed by Ky Fan's Inequality (Theorem 14). Let $\{p^\nu \in \Sigma, \nu \in N \subset I\!N\}$ be a converging subsequence of such equilibrium prices; such a converging subsequence exists since Σ is compact. Let $p^\nu \xrightarrow[N]{} \bar{p}$ be the limit point. Of course, $\bar{p} \in \Sigma$. To prove that $\bar{p}$ is an equilibrium point. We proceed as follows: For all $\nu \in N$, $s^\nu(p^\nu) \ge 0$ since p^ν is an equilibrium price for $\mathcal{E}^\nu$; recall that $s^\nu(\cdot)$ denotes the excess supply function for $\mathcal{E}^\nu$. Restricting ourselves to the subsequence $N \subset I\!N$, it follows from the hypotheses and Corollary 6 that these functions s^ν continuously converge to s, the excess supply function of $\mathcal{E}$. So, in particular, $s^\nu(p^\nu) \xrightarrow[N]{} s(\bar{p})$, and since for all $\nu \in N$, $s^\nu(p^\nu) \ge 0$, one has $s(\bar{p}) \ge 0$, i.e., $\bar{p}$ is an equilibrium point.

This is essentially the argument used by Lucchetti and Patrone [15, 12]. It shortcuts the need to introduce the Walrasian and lopsided convergence, but as already pointed out in the Introduction, one doesn't then arrive at the concept of 'converging economies.'

5. An example

To illustrate how our main result can be applied, let's consider the following variation of an example of Mas-Collel *et al* [17, Section 15.B]. In this economy $\mathcal{E}$, two goods are being traded by two agents with strictly concave utility functions:

$$u_1(x_{11}, x_{21}) := \begin{cases} x_{11} - (1/8)x_{21}^{-8}, & \text{on } [0.3, 3]^2 \ , \\ -\infty & \text{otherwise} \ , \end{cases}$$

$$u_2(x_{12}, x_{22}) := \begin{cases} x_{22} - (1/8)x_{12}^{-8}, & \text{on } [0.3, 3]^2 \ , \\ -\infty & \text{otherwise} \ , \end{cases}$$

where x_{li} refers to the amount of good l consumed by the agent i. Survival requires a minimal consumption of 0.3 units of each good and there is no interest in consumption exceeding 3 units. Thus the domain for u_1 and u_2 are respectively:

$$\operatorname{dom} u_1 = \operatorname{dom} u_2 = [0.3, 3] \times [0.3, 3].$$

The initial endowments are $e_1 = (2, r)$ and $e_2 = (r, 2)$, with $r = 2^{8/9} - 2^{1/9}$ chosen so as to obtain nice numbers for the equilibrium points. Thus, the initial endowments are in $\operatorname{int}(\operatorname{dom}\ u_1)$ and $\operatorname{int}(\operatorname{dom}\ u_2)$ respectively.

Next, we compute the equilibrium points. For a price vector (p_1, p_2) such that each component is nonzero, the utility maximization problem subject to the budget constraint gives us the following results; observe that the maximization points end up in $\operatorname{int}(\operatorname{dom}\ u_i)$. For agents 1 and 2, one obtains,

$$x_{21} = (p_2/p_1)^{-1/9}, \quad x_{11} = 2 + (r - (p_2/p_1)^{-1/9})p_2/p_1$$

and

$$x_{12} = (p_1/p_2)^{-1/9}, \quad x_{22} = 2 + (r - (p_1/p_2)^{-1/9})p_1/p_2.$$

So, from the equation 'supply equals demand' applied to the second good, one deduces

$$(p_2/p_1)^{-1/9} + 2 + (r - (p_1/p_2)^{-1/9})p_1/p_2 = 2 + r.$$

In this example, this last equation suffices to compute all the equilibrium points. The three solutions of this equation are $p_1/p_2 = 0.5, 1, 2$ that, in turn, yield three equilibrium points (in Σ).

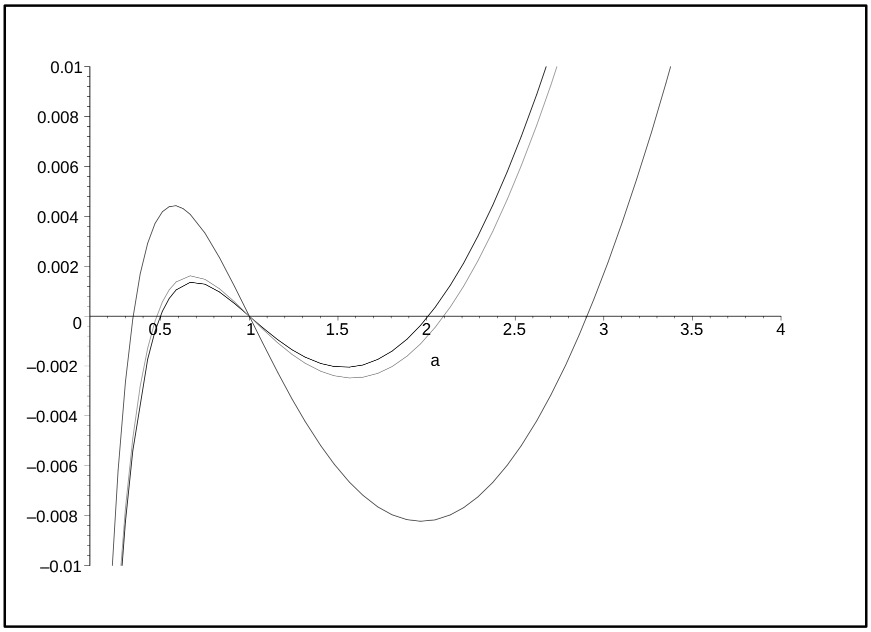

Figure 2: Convergence of equilibrium points

In order to analyze the behavior of equilibrium points under perturbations, we considered two types of perturbations for one of the utilities functions, namely, linear and scaling perturbations for u_1:

$$\text{case 1:} \quad u_1^\nu(x_{11}, x_{21}) := \begin{cases} x_{11} - (1/8)x_{21}^{-8} + x_{11}/\nu & \text{on } [0.3, 3]^2 \ , \\ -\infty & \text{otherwise} \ , \end{cases}$$

$$\text{case 2:} \quad u_1^\nu(x_{11}, x_{21}) := \begin{cases} x_{11} - (1/8)x_{21}^{-8}(1 + 1/\nu) & \text{on } [0.3, 3]^2 \ , \\ -\infty & \text{otherwise} \ , \end{cases}$$

The main result of this paper can now be applied to this example, indeed, the dom $u_1^\nu =$ dom u_1 for all ν and continuous convergence of u_1^ν to u_1 is obtained from the concavity of u_1^ν and u_1 and the compactness of their domains [18, Theorem 7.17]. Thus, for these perturbations we conclude that, the clusters points of the sequences p_1^ν/p_2^ν are also equilibrium prices.

On the other hand, computing the new equilibrium points in terms of ν –we relied on MAPLE– for the solutions of the supply-demand equations, it is possible to obtain an explicit verification of the implications of the theorem. Indeed, in both cases, $p_1^\nu/p_2^\nu = 1$ is, for all ν, a solution of the 'supply equals demand' equation. And, there are two more sequences of solutions: for the first one has $p_1^\nu/p_2^\nu \nearrow 0.5$ and for the second $p_1^\nu/p_2^\nu \searrow 2$ as $\nu \to \infty$.

In Figure 2, the x-axis is p_1^ν/p_2^ν and y-axis is the excess demand. The various curves correspond to the cases $\nu = 10, 100, 1000$.

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