Calculus 16B, Spring '07, Prof Sims
HW# 2 Solutions
(by TA David Cherney)

§4.6

6) $y = Ce^{kt}$ passes through the points $(t, y) = (3, 1/2)$ and $(t, y) = (4, 5)$ so the following two equations hold
$\frac{1}{2} = Ce^{k3}$, $5 = Ce^{k4}$
We now have two equations and two unknowns, so we can solve for both k and C. Solving the first equation for C gives $C = \frac{1}{2}e^{-3k}$. Substituting into the second equation gives $5 = \frac{1}{2}e^{-3k}e^{4k} = \frac{1}{2}e^{4k-3k} = \frac{1}{2}e^{k}$. Multiply both sides by two and take the natural log of both sides to get $\ln(10) = k$. Lasly, substitute this back into $C = \frac{1}{2}e^{-3k}$ to obtain $C = \frac{1}{2}e^{-3\ln(10)}$ and simplify to get $C = \frac{1}{2}e^{\ln\left(10^{-3}\right)} = \frac{1}{2}10^{-3}$. The equation is then
$y = Ce^{kt} = \frac{1}{2}10^{-3}e^{\ln(10)t} = \frac{1}{2}10^{-3}e^{\ln\left(10^{t}\right)} = \frac{1}{2}10^{-3}10^{t} = \frac{1}{2}10^{t-3}$.

8) We know that the derivative of $y = Ce^{kt}$ is $y' = k\left(Ce^{kt}\right)$, so if $y' = -\frac{2}{3}y$ then we can conclude $k = -\frac{2}{3}$. Then $y = Ce^{-\frac{2}{3}t}$ and plugging in the point $(t, y) = (0, 20)$ we get $20 = Ce^{-\frac{2}{3}0} = C$. The function is then $y = 20e^{-\frac{2}{3}t}$.

20) The amount of ^{14}C in the chracoal has been decreasing exponentialy since it was created (by bruning.) The amount of ^{14}C left t years after the creation of the chrcoal is given by $y = Ce^{kt}$ for some k and some C. We need to figure out what these values are. First notice that, since we know the half life, and we know that C is the amount of C^{14} at $t = 0$, we know that $\frac{1}{2}C = Ce^{k5715}$ which we can rearange to find k (but tells us nothing about C singe C just cancels out):
$\frac{1}{2} = e^{k5715} \Rightarrow \ln\left(\frac{1}{2}\right) = 5715k \Rightarrow k = \frac{1}{5715}\ln\left(\frac{1}{2}\right) = \frac{1}{5715}\ln\left(2^{-1}\right)$
$= -\frac{1}{5715}\ln(2)$
Notice that k is negative so we can see in the equation $y = Ce^{kt}$ that y decreases in time (if you cant picture the graph just by looking at the equation you can at least see that the derivative is negative since C, the initial amount, is going to be positive.)

Now, the question was "how long ago was a piece of charcoal with 30% as much ^{14}C as a modern piece created.. so we need to solve $0.3C = Ce^{kt}$ for t and then substitute in what we know k to be from the discussion of the half life above: $t = \frac{1}{k}\ln(0.3) = -\frac{5715}{\ln(2)}\ln(0.3) = 9,926$. It makes sense that this is longer than the half life since more than half of the ^{14}C has decayed, namely 30% has decayed. (Notice that I wrote the symbol k untill the very last step where I substituted in its numerical value. This is a good way to do things in general since you have to write less, it looks neater, you spend less time on your calculator by saving all the calculations for the end and you are more likely to catch yourself making a mistake. I highly recomend it.)

41) Using the given equation and information we can say $19 = 30\left(1 - e^{k20}\right) \Rightarrow 30e^{20k} = 11 \Rightarrow 20k = \ln\left(\frac{11}{30}\right) \Rightarrow k = \frac{1}{20}\ln\left(\frac{11}{30}\right)$ and we want to solve for t in the equation $25 = 30\left(1 - e^{kt}\right) \Rightarrow e^{kt} = \frac{5}{30} \Rightarrow t = \frac{1}{k}\ln\left(\frac{5}{30}\right) = \frac{20}{\ln\left(\frac{11}{30}\right)}\ln\left(\frac{5}{30}\right)$ =35.7...days. This is longer than 20 days, and it makes sense that it would take longer to learn to make 25 units than it would to make 19 units. (always do a "sanity check" at the end of your word problems! You will catch almost all you mistakes by simply asking yourslef "Does this makes sense?")

44) We have the two equations $45 = Ce^{k1000}$ and $40 = Ce^{k1200}$. Divide them to get $\frac{40}{45} = e^{200k} \rightarrow k = \frac{1}{200}\ln\left(\frac{40}{45}\right)$. Substitute this into $45 = Ce^{k1000}$ to get

$45 = Ce^{\frac{1}{200}\ln\left(\frac{40}{45}\right)1000} = Ce^{5\ln\left(\frac{40}{45}\right)} = Ce^{\ln\left(\frac{40}{45}\right)^5} = C\left(\frac{40}{45}\right)^5 \Rightarrow C = 45\left(\frac{45}{40}\right)^5$.

48) a) $R = 8.3 = \frac{\ln(I) - \ln(I_0)}{\ln(10)} \Rightarrow \ln(I) - \ln(I_0) = 8.3\ln(10) = \ln(10)^{8.3}$

$\Rightarrow I(R) = I_0 10^{8.3} = 10^{8.3} (= 10^R)$

b) $\frac{I(2R)}{I(R)} = \frac{10^{2R}}{10^R} = 10^R$

c) $\frac{dR}{dI} = \frac{1}{\ln(10)}\frac{d}{dI}\ln(I) = \frac{1}{I\ln(10)}$

§5.1

8) On the left side, the derivative of the antiderivative of a function is just the function $\frac{d}{dx}\int \frac{2x-1}{x^{4/3}}dx = \frac{2x-1}{x^{4/3}}$ and on the other side I'll use the

product rule (you can use the quotent rule too) $\frac{d}{dx}\left[\frac{3\,(x+1)}{x^{1/3}}+C\right]=\frac{3}{x^{1/3}}-\frac{1}{3}\frac{3\,(x+1)}{x^{4/3}}=\frac{3x-x-1}{x^{4/3}}.$

20) $\int \nu^{-1/2}d\nu = 2\nu^{1/2}+c.$

30) The function $y=-2x$ has slope -2 everywhere, so pretend I sketched it.

32) This graph shows the function $y=-x$, a function with antderivative $y=-\frac{1}{2}x^2$. Pretend I sketched it.

36) $\int x^{1/2}+\frac{1}{2}x^{-1/2}dx=\frac{2}{3}x^{3/2}+x^{1/2}+C.$

38) $\int x^{3/4}+1dx=\frac{4}{7}x^{7/4}+x+C$

56) $y=x^2-2x+c$. We know the point $(x,y)=(3,2)$ lies on the graph of this function so we know $2=3^2-2\cdot 3+c=9-6+c=3+c\Rightarrow c=-1$. The particular solution which passes through $(x,y)=(3,2)$ is then $y=x^2-2x-1$.

60) $f'(x)=\frac{1}{3}x^3+C$ and we know $f'(0)=C=6$. Antidifferentiating again $f(x)=\frac{1}{12}x^4+6x+K$ and we know $f(0)=K=3$ so $f(x)=\frac{1}{12}x^4+6x+3$.

74) $P(x)=-15x^2+920x+C$ and $P(8)=-15\cdot 8^2+920\cdot 8+C=6500\rightarrow C=6500+15\cdot 8^2-920\cdot 8=100$. The profit function is then $P(x)=-15x^2+920x+100$.

78) a) The acceleration of the sandbag due to gravity is $x''(t)=-32\frac{ft}{s^2}$ negative because gravity pulls down) so that it's velocity is given by $x'(t)=-32t+v_0$, but we know that the velocity of the bag at $t=0$ is the velocity of the baloon which was carrying it, namely $16\frac{ft}{s}$, so $x'(t)=-32t+16$. Antidiferentiating again shows that the height is given by $x(t)=-16t^2+16t+x_0$, but we know that the height at $t=0$ is 64 feet so the height is given by $x(t)=-16t^2+16t+64$. The bag hits the ground when $x=0$, and so we must solve $-16t^2+16t+64=0$. Divide through by -16 to get $t^2-t-4=0$ and then complete the square or use the quadratic formula

to get $t = \frac{1}{2} \pm \frac{1}{2}\sqrt{17}$. The solution with the minus makes no sense because ut is less than zero, surely the bag hit the ground *after* it was released, not beofre.

b) now we simply plug the time at which the bag hits the ground into the velocity function
$x'\left(\frac{1}{2}\left(1+\sqrt{17}\right)\right) = -32\frac{1}{2}\left(1+\sqrt{17}\right)+16 = -16\left(1+\sqrt{17}\right)+16$
$= -16\sqrt{17}\frac{ft}{s}$.

§5.2
14) $\frac{2}{7}(x-3)^{7/2}+C$, 16) $-\frac{1}{4}\frac{1}{4}\left(1-2x^2\right)^4+C$, 22) $-\left(x^3+3x+7\right)^{-2}+C$, 26) $-\frac{1}{3}2\left(1-x^3\right)^{1/2}+C$, 38) $\frac{2}{3}\frac{1}{2}\left(t^2+1\right)^{3/2}+C$, 42) $-\frac{1}{3}\frac{2}{3}\left(4-x^{3/2}\right)^3+C$
48) $f(x) = -\frac{2}{3}\frac{1}{2}\left(1-x^2\right)^{3/2}+C$. But $\frac{7}{3} = 0+C$ so the function is $f(x) = -\frac{1}{3}\left(1-x^2\right)^{3/2}+\frac{7}{3}$.