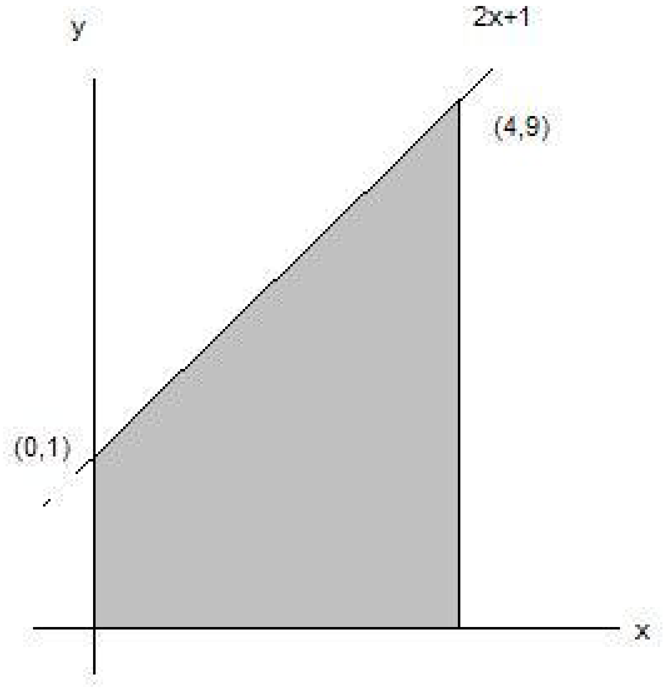
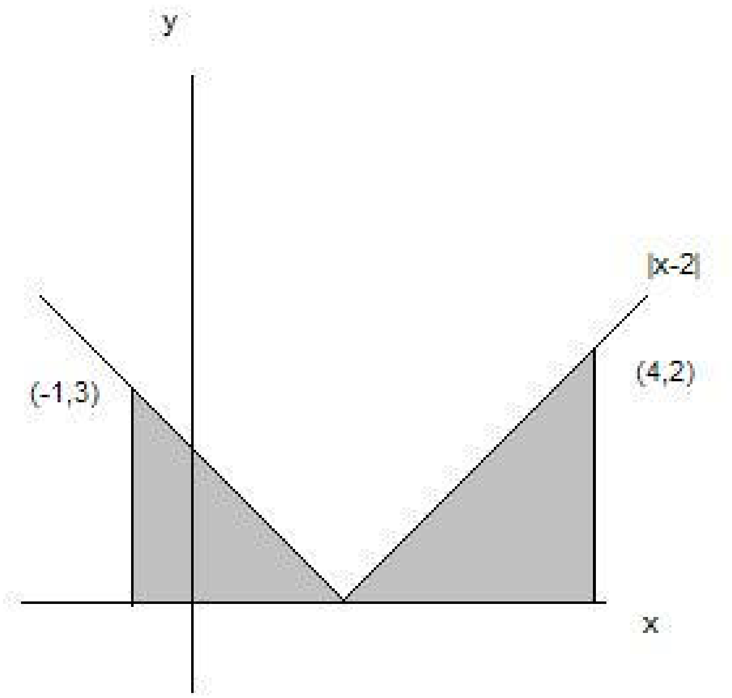
Calculus 16B, Spring '07, Prof Sims
HW# 2 Solutions
(by TA David Cherney)

§5.4
4) My sketch:



This is part of a right triangle with height 9 and base $4 + 1/2$ with the left-most tip, a right triangle with height 1 and base 1/2, cut off. The area of a triangle is $1/2 \times \text{base} \times \text{height}$. Thus the area is $\frac{1}{2} 9 \frac{9}{2} - \frac{1}{2} 1 \frac{1}{2} = \frac{81-1}{4} = 20$, so this is the value of the integral.

6) Pardon my not to scale drawing (Some things will always be easier with a pencil.)



The area cosists of two triangles, one with base 3 and height 3, the other with base 2 and height 2, so the value of the integral is $\frac{1}{2}2 \cdot 2 + \frac{1}{2}3 \cdot 3 = 2 + 9/2 = 13/2$.

10) a. $\int_0^5 2g(x)\,dx = 2\int_0^5 g(x)\,dx = 2(8) = 16$

b. $\int_5^5 f(x)\,dx = 0$ since if we could find an antiderivative F of f then $\int_5^5 f(x)\,dx = [F(x)]_5^5 = F(5) - F(5) = 0$.

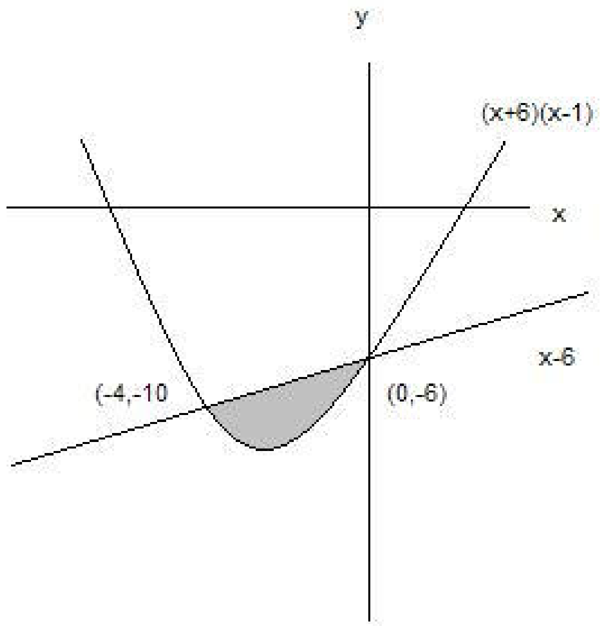
14) This is a pretty strange reagon, so we better use an integral. $4\sqrt{x}$ is an antiderivative of the integrand since $\frac{d}{dx}4\sqrt{x} = 4\frac{d}{dx}x^{1/2} = 4\frac{1}{2}x^{-1/2} = \frac{2}{\sqrt{x}}$. So, $\int_1^4 \frac{2}{\sqrt{x}}dx = \left[\sqrt{x}\right]_1^4 = 2 - 1 = 1$, not a very strange number!

26) $\int_2^2 (x-3)^4\,dx = 0$ for the same reason as in # 10b.

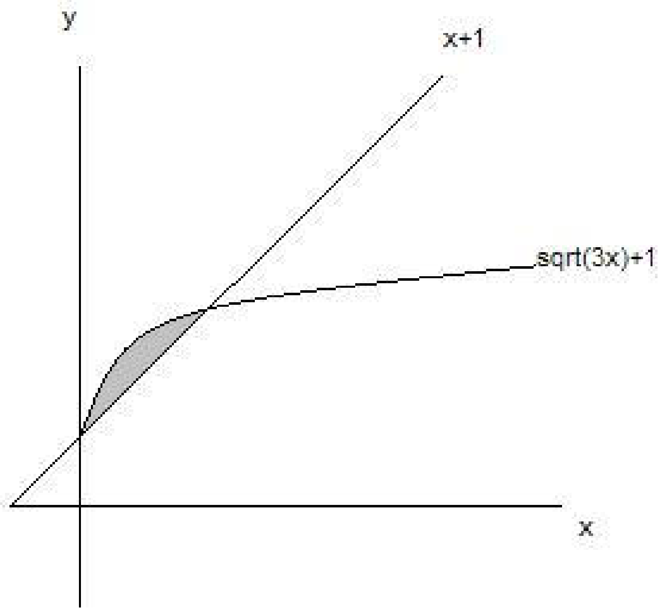
32) $\int_0^4 \left(x^{1/2}+x^{1/4}\right)dx = \int_0^4 x^{1/2}dx + \int_0^4 x^{1/4}dx$
$= \left[\frac{2}{3}x^{3/2}\right]_0^4 + \left[\frac{4}{5}x^{5/4}\right]_0^4 = \frac{2}{3}4^{3/2} + \frac{4}{5}4^{5/4}$
$= \frac{2}{3}2^3 + \frac{4}{5}2^{5/2} = \frac{16}{3} + \frac{2^{7/2}}{5}$
36) $\int_1^2 e^{(1-x)}dx = e\int_1^2 e^{-x}dx = e\left[-e^{-x}\right]_1^2$
$= e\left(-e^{-2}+e^{-1}\right) = 1-e^{-1}$
44) $|2x-3| = 2x-3$ for $x > 3/2$, $|2x-3| = -2x+3$ for $x < 3/2$
$\int_0^3 |2x-3|\,dx = \int_0^{3/2}(-2x+3)\,dx + \int_{3/2}^3 (2x-3)\,dx$
$= \left[-x^2+3x\right]_0^{3/2} + \left[x^2-3x\right]_{3/2}^3 = -2\left(\frac{3}{2}\right)^2 + 2\cdot 3\frac{3}{2} + 3^2 - 3\cdot 3$
$= -\frac{9}{2} + 9 = \frac{9}{2}$. It would be easyer to graph this function and look at the area under the curve, it consists of some triangles.
68) The average is $\frac{1}{1-0}\int_0^1 \frac{4x}{x^2+1}dx = \left[2\ln\left|x^2+1\right|\right]_0^1 = 2\ln(2) - 0$. The function attains this value when $\frac{2x}{x^2+1} = \ln(2)$. This means $\ln(2)\,x^2 - 2x + \ln(2) = 0$. Using the quadratic formula the solutions are $x = \frac{2\pm\sqrt{4-4\ln^2(2)}}{2\ln(2)} = \frac{1\pm\sqrt{1-\ln^2(2)}}{\ln(2)}$.
70) $g(-x) = (-x)^3 - 2(-x) = -g(x)$ so this is an odd function.
74) I'll start with the fact that x^2 is an even function so
$\int_{-2}^2 x^2dx = 2\int_0^2 x^2dx = 2\cdot 8/3$
$= \int_{-2}^0 x^2dx + \int_0^2 x^2dx = \int_{-2}^0 x^2dx + 8/3$
So $\int_{-2}^0 x^2dx = 8/3$.
82) $500e^{.09\cdot 4}\int_0^4 e^{-.09t}dt = 500e^{.09\cdot 4}\left[-\frac{1}{.09}e^{-.09t}\right]_0^4$
$= 500e^{.09\cdot 4}\left(-\frac{1}{.09}e^{-.09\cdot 4} + \frac{1}{.09}\right) = \frac{500}{.09}\left(e^{.09\cdot 4}-1\right)$

§5.5

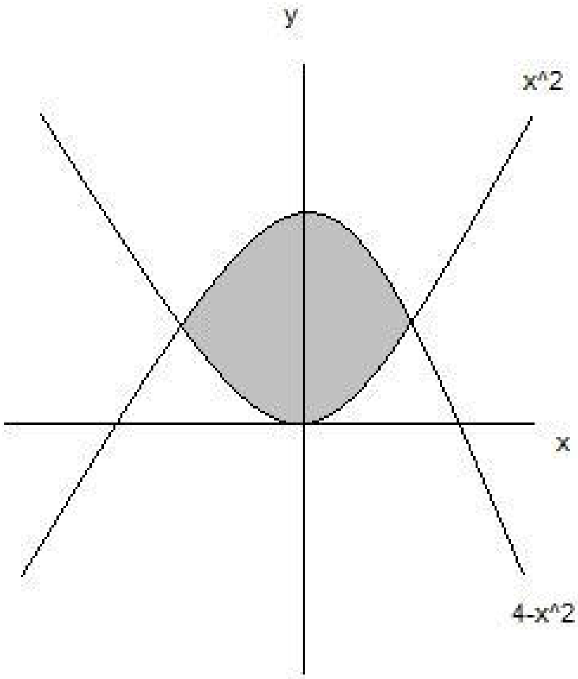
6) $\int_0^1 (x-1)^3 - (x-1)\,dx + \int_1^2 (x-1) - (x-1)^3\,dx$
$= \left[\frac{1}{4}(x-1)^4 - \frac{1}{2}(x-1)^2\right]_0^1 + \left[\frac{1}{2}(x-1)^2 - \frac{1}{4}(x-1)^4\right]_1^2$
$= -\dfrac{1}{4} + \dfrac{1}{2} + \dfrac{1}{2} - \dfrac{1}{4} = 1 - \dfrac{1}{2} = \dfrac{1}{2}.$
8) $\int_1^2 \left(-x + 3 - \dfrac{2}{x}\right) dx = \left[-x^2/2 + 3x - 2\ln(x)\right]_1^2$
$= -4/2 + 3 \cdot 2 - 2\ln(2) + 1/2 - 3$
$= -2 + 6 - 2\ln(2) + 1/2 - 3$
$= 3/2 - 2\ln(2)$
12) $x^2 + 5x - 6 = (x+6)(x-1)$ and that gives the x-intercepts $x = -6, 1$. I use that and the positive sign on x^2 to plot this function. To see where the two functions intersect,
$x^2 + 5x - 6 = x - 6 \Rightarrow x^2 + 4x = 0 \Rightarrow x = 0, -4$, and of course $x - 6$ is a linear function. Put it all together and...



18) These functions are just $\sqrt{3x}$ and x shifted up by one. These intersect at $x = 0, 3$.



The area inclosed is
$\int_0^3 \left(\sqrt{3x}+1\right) - (x+1)\,dx = \int_0^3 \left(\sqrt{3x}-x\right)dx$
$= \left[\sqrt{3}\frac{2}{3}x^{3/2} - \frac{x^2}{2}\right]_0^3 = \sqrt{3}\frac{2}{3}3^{3/2} - \frac{3^2}{2} = 2\cdot 3 - \frac{9}{2}$
20) The functions intersect at $x = \pm\sqrt{2}$ and are parabolas.



The area enclosed

$$\int_{-\sqrt{2}}^{\sqrt{2}} \left(4 - x^2\right) - x^2 dx = 2\int_0^{\sqrt{2}} \left(4 - x^2\right) - x^2 dx$$

$$= 2\int_0^{\sqrt{2}} \left(4 - 2x^2\right) dx = 4\left[2x - \frac{1}{3}x^3\right]_0^{\sqrt{2}}$$

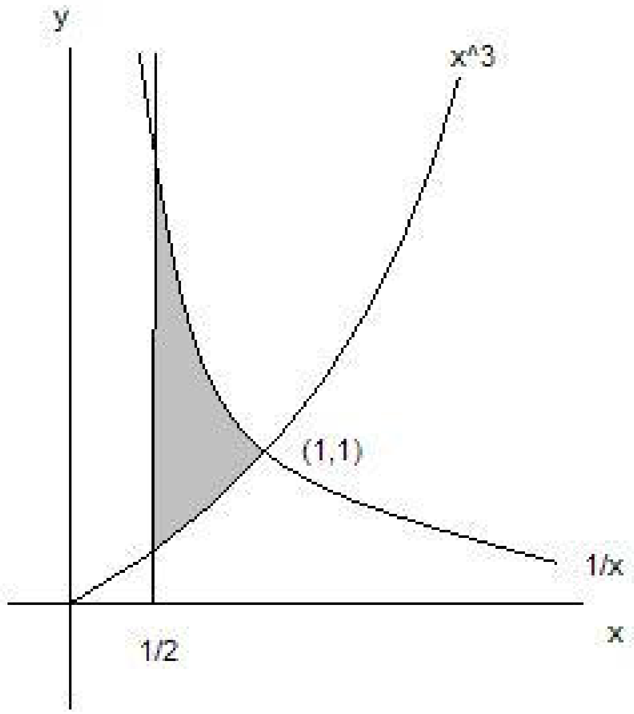
$$= 4\left(2\sqrt{2} - \tfrac{1}{3}2^{3/2}\right) = 8\sqrt{2}\left(1 - \tfrac{1}{3}\right)$$

$$= 8\sqrt{2}\frac{2}{3}$$

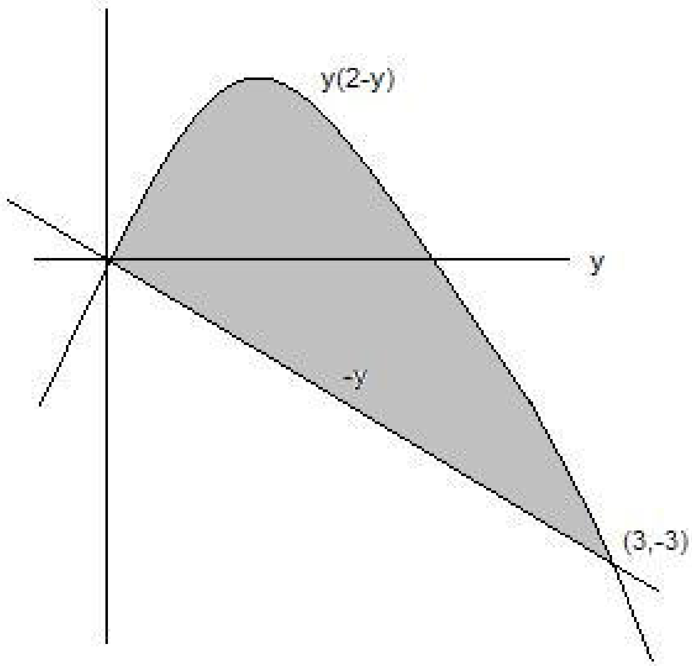
22) I have no idea what $y = \dfrac{e^{1/x}}{x^2}$ looks like, but I do see that its positive everywhere, and the region in question is bounded below by $y = 0$. I had to stare at this for a while before I saw what the antiderivative was.

$$\int_1^3 \frac{e^{1/x}}{x^2} dx = \left[-e^{1/x}\right]_1^3 = e - e^{1/3}$$

24) The two functions equal eachother when $x^4 = 1 \Rightarrow x = \pm 1$.

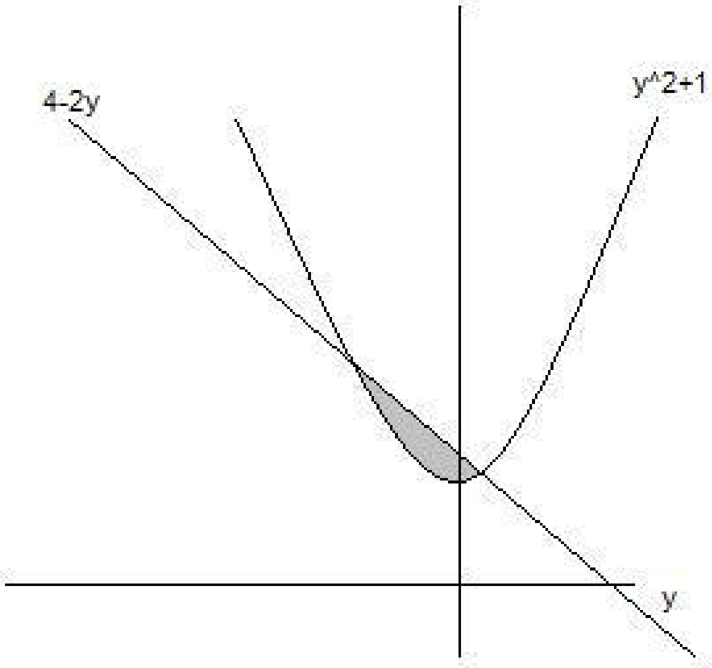


The area enclosed is
$\int_{1/2}^{1} \frac{1}{x} - x^3 dx = \left[\ln|x| - \frac{x^4}{4}\right]_{1/2}^{1} = -\frac{1}{4} - \ln(1/2) + \frac{(1/2)^4}{4}$
$= \ln(2) + \frac{1}{64} - \frac{1}{4}$.
28) The functions intersect when $2y - y^2 = -y \Rightarrow y(3-y) = 0$ or when $y = 0, 3$.



The area between the functions is
$\int_0^3 \left(2y - y^2\right) - (-y)\, dy = \left[y^2 - \frac{1}{3}y^3 + \frac{1}{2}y^2\right]_0^3 = \frac{3}{2}3^2 - \frac{1}{3}3^3$
$= \frac{3^3}{2} - 3^2.$

30) The functions intersect when $y^2 + 1 = 4 - 2y \Rightarrow (x+3)(x-1) = 0$, so when $x = -3, 1$.



The area enclosed is

$$\int_{-3}^{1} (4-2y) - \left(y^2+1\right) dx = \int_{-3}^{1} \left(3-2y-y^2\right) dy$$

$$= \left[3y - y^2 - \frac{1}{3}y^3\right]_{-3}^{1} = 3 - 1 - \frac{1}{3} - 3\cdot(-3) + 3^2 + \left(-\frac{1}{3}(-3)^3\right)$$

$$= 2 - \frac{1}{3} + 3^2 + 3^2 + 3^2 = 2 - \frac{1}{3} - 3^3$$

§5.7

2) The disk at x has radius $r = x^2$ and volume $\pi r^2 dx = \pi \left(x^2\right)^2 dx$. Summing over these disks from $x = 0$ to $x = 1$ gives

$$\int_0^1 \pi x^4 dx = \frac{\pi}{5}.$$

4) The disk at x has radius $r = \sqrt{4-x^2}$ and volume $\pi r^2 dx = \pi\left(4-x^2\right) dx$. Summing over these gives

$$\int_{-2}^{2} \pi\left(4-x^2\right) dx = 2\pi \int_0^2 \left(4-x^2\right) dx$$

$$= 2\pi\left(4x - \tfrac{1}{3}x^3\right)_0^2 = 2\pi\left(8-\tfrac{8}{3}\right)$$

14) The disk at x has radius $r = \dfrac{1}{x}$ and volume $\pi r^2 dx = \pi \dfrac{1}{x^2} dx$. Summing over these gives

$\pi \int_1^3 \frac{1}{x^2} dx = \pi \left[-\frac{1}{x} \right]_1^3 = \pi \left[1 - \frac{1}{3} \right] = \frac{2\pi}{3}$

16) These functions intersect when $x^2 = 4x - x^2 \Rightarrow x(2-x) = 0 \Rightarrow x = 0, 2$ and $4x - x^2$ is the greater function in the region in between these values. The area enclosed is entirely above the x axis, since x^2 is positive everywhere. The radius of a disk at x is $r_{out} = (4x - x^2)$, but we must subtract the region missing in each disk, a region with radius $r_{in} = x^2$. So we end up with washer shaped pieces of volume $\pi r_{out}^2 dx - \pi r_{in}^2 dx = \pi \left[(4x - x^2)^2 - (x^2)^2 \right] dx$. Summing over these gives

$\int_0^2 \pi \left[\left(4x - x^2\right)^2 - x^4 \right] dx = \int_0^2 \pi \left[16x^2 + x^4 - 8x^3 - x^4 \right] dx =$

$\pi \left[\frac{16}{3} x^3 - \frac{8}{4} x^4 \right]_0^2 = \pi \left[\frac{16}{3} 8 - \frac{8}{4} 16 \right]$

18) I will use a trick on this one: $y = \sqrt{4^2 - x^2}$ is the equation of a circle of radius 4 centered at the origin, and the range of x reduces us to a quartercircle in the first quadrant. Rotating this around the y-axis gives a hemi-sphere of radius 4, which of course has volume $\frac{1}{2} \pi r^3 = \frac{1}{2} \cdot \frac{4}{3} \pi 4^3$.

20) Dont let this $x(y)$ way of writing things confuse you. Just draw a graph with the vertical axis labeled x-axis and the horizontal axis labeled y-axis. It is obvious that $y(y-1) = 0$ when $y = 0, 1$. If you want to draw it like the book, just turn your paper over and rotate it clockwise 90 degrees and trace what you just did. Either way, the radius of the disk at y is $r = y(y-1)$ and its volume is $\pi r^2 dy = \pi \left(y^2 - 1\right)^2 dy$. Summing over these we get

$\int_0^1 \pi \left(y^4 + 1 - 2y^2\right) dx = \pi \left[\frac{1}{5} + 1 - \frac{2}{3} \right]$

22) Factor out one y to get $x = y(4-y)$ which is zero when $y = 0, 4$. The radius of the disk at y is $r = y(4-y)$ and its volume is $\pi r^2 dy = \pi \left(16y^2 + y^4 - 2 \cdot 4y^3\right) dy$. Summing these we get

$\int_0^4 \pi \left(16y^2 + y^4 - 2 \cdot 4y^3\right) dy = \left[\frac{16}{3} 4^3 + \frac{1}{5} 4^5 - 24^3 \right]$

26) Lets do this problem with disks around the y axis. The line segment lies on the the line $y = \frac{2}{4} x + 0 \Rightarrow x = 2y$. Thus, the radius of the disk at y is $2y$ and its volume is $\pi 4y^2 dy$. Summing over these from $y = 0$ to $y = 2$ gives

$\int_0^2 \pi 4y^2 dy = \frac{4\pi}{3} 2^3.$

28) We can get a sphere by rotating the semicircle $y = \sqrt{r^2 - x^2}$ about the x axis. The radius of the disk at x is $r = \sqrt{r^2 - x^2}$ and its volume is $\pi r^2 dx = \pi \left(r^2 - x^2\right) dx$. Summing over these from $x = -r$ to $x = r$ gives

$\int_{-r}^r \pi \left(r^2 - x^2\right) dx = \int_0^r 2\pi \left(r^2 - x^2\right) dx = 2\pi \left[r^2 x - \frac{1}{3} x^3 \right]_0^r$

$= 2\pi \left(r^3 - \frac{1}{3} r^3 \right) = \frac{4}{3} \pi r^3.$

Which was to be shown (or, in Latin, quod erat demonstrandum or QED for nerdy short.)

30) Solving for y I get $y = \pm 3\sqrt{1 - \frac{x^2}{16}}$. I pick the function with the + sign (either one, when rotated, will give the same volume.) The radius of the disk at x is $r = 3\sqrt{1 - \frac{x^2}{16}}$ so its volme is $\pi r^2 dx = \pi 9 \left(1 - \frac{x^2}{16}\right) dx$. The zero's of my function are clearly ± 4, so I add up all the disks in the region between these values to get

$\int_{-4}^{4} \pi 9 \left(1 - \frac{x^2}{16}\right) dx = \int_{0}^{4} 18\pi \left(1 - \frac{x^2}{16}\right) dx$

$= 18\pi \left(4 - \frac{4^3}{3 \cdot 4^2}\right) = 18\pi 4 \left(\frac{2}{3}\right).$