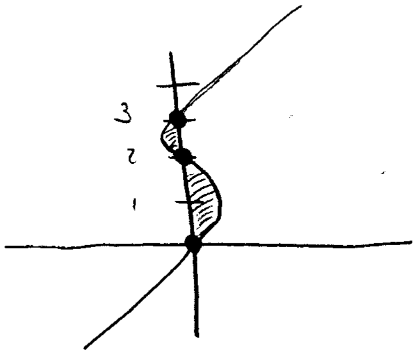
Fake Test 2B (Key)

1) (a)



$$f(y) = y^3 - 5y^2 + 6y$$
$$= y(y^2 - 5y + 6)$$
$$= y(y-2)(y-3)$$

$$\text{Area} = \int_0^2 [y^3 - 5y^2 + 6y] - 0\,dy + \int_2^3 0 - [y^3 - 5y^2 + 6y]\,dy$$

$$= \left(\frac{y^4}{4} - \frac{5y^3}{3} + \frac{6y^2}{2}\right)\Bigg|_0^2 + \left(-\frac{y^4}{4} + \frac{5y^3}{3} - \frac{6y^2}{2}\right)\Bigg|_2^3$$

$$= \left(4 - \frac{40}{3} + 12\right) + \left(-\frac{81}{4} + 45 - 27\right) - \left(-4 + \frac{40}{3} - 12\right)$$

$$= 32 - \frac{80}{3} - \frac{81}{4} + 18$$

$$= 32 - 26\tfrac{2}{3} - 20\tfrac{1}{4} + 18$$

$$= 4 - \frac{2}{3} - \frac{1}{4} = \frac{48}{12} - \frac{8}{12} - \frac{3}{12} = \frac{37}{12}$$

(b) The average value of f on $[0,3]$ is

$$\frac{1}{3}\int_0^3 y^3 - 5y^2 + 6y\,dy = \frac{1}{3}\left[\frac{y^4}{4} - \frac{5y^3}{3} + \frac{6y^2}{2}\right]\Bigg|_0^3$$

$$= \frac{1}{3}\left(\frac{81}{4} - 45 + 27\right) = \frac{9}{12} = \frac{3}{4}$$

(c) In part a) we calculated the area of the regions between the curve and the y-axis. In part b) we merely integrated the function and divided by the size of the interval.

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(2) (a) If f is odd and the average value of f on $[0,5]$ is 10, then

$$\frac{1}{5}\int_0^5 f(x)\,dx = 10 \quad \text{i.e.} \quad \int_0^5 f(x)\,dx = 50.$$

Since f is odd,

$$\int_{-5}^0 f(x)\,dx = -\int_0^5 f(x)\,dx = -50.$$

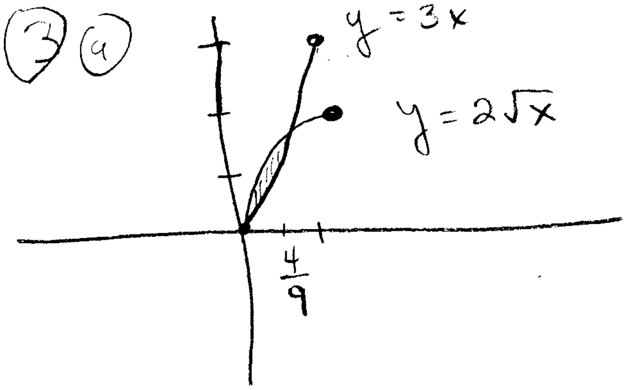
(b) If g is even and the average value of g on $[-3,3]$ is 9, then.

$$\frac{1}{6}\int_{-3}^3 g(x)\,dx = 9 \quad \text{i.e.} \quad \int_{-3}^3 g(x)\,dx = 54$$

As g is even,

$$\int_0^3 g(x)\,dx = \frac{1}{2}\int_{-3}^3 g(x)\,dx = \frac{54}{2} = 27$$

3 (a)



Where do they meet?

$$3x = 2\sqrt{x}$$

$$9x^2 = 4x$$

$$9x^2 - 4x = 0$$

$$x(9x-4) = 0$$

$$x = 0\ ,\ \frac{4}{9}$$

(b)

$$\text{Volume} = \pi \int_0^{4/9} (2\sqrt{x})^2 - (3x)^2\, dx$$

$$= \pi\left(4 \int_0^{4/9} x\, dx - 9 \int_0^{4/9} x^2\, dx \right)$$

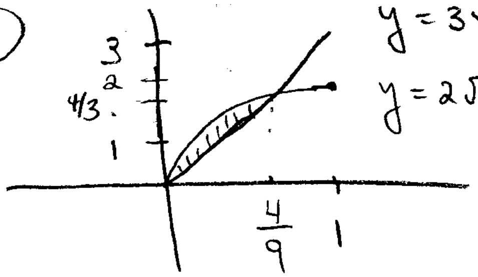
$$= \pi\left[2x^2 \Big|_0^{4/9} - 3x^3 \Big|_0^{4/9} \right]$$

$$= \pi\left(2\cdot\frac{4^2}{9^2} - 3\,\frac{4^3}{9^3} \right)$$

$$= \frac{4^2}{9^2}\pi\left(2 - \cancel{3}\frac{4}{\cancel{9}3} \right)$$

$$= \frac{16}{25}\cdot\frac{2}{3}\pi = \frac{32}{75}\pi$$

(c)



if $y = 3x$, then $x = \frac{y}{3}$

if $y = 2\sqrt{x}$, then $x = \frac{y^2}{4}$

at $x = 4/9$, $y = 3(4/9) = \frac{4}{3}$

hus

$$\text{Volume} = \pi \int_0^{4/3} \left(\frac{y}{3}\right)^2 - \left(\frac{y^2}{4}\right)^2 dy$$

$$= \pi \left[\int_0^{4/3} \frac{y^2}{3^2} dy - \int_0^{4/3} \frac{y^4}{4^2} dy \right]$$

$$= \pi \left[\frac{y^3}{3^3} \Big|_0^{4/3} - \frac{y^5}{4^2 \cdot 5} \Big|_0^{4/3} \right]$$

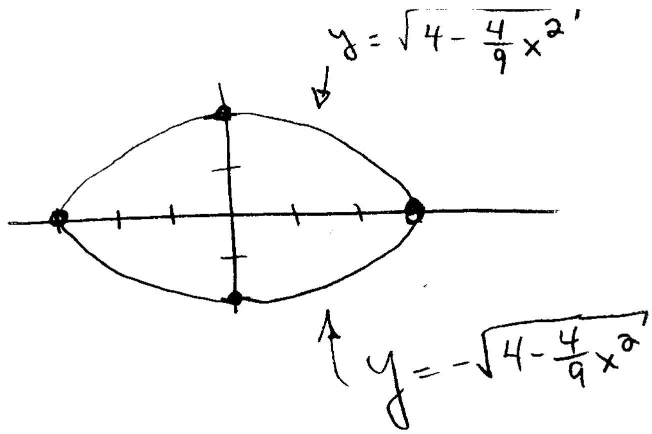
$$= \pi \left[\frac{4^3}{3^3 \cdot 3^3} - \frac{4^5}{4^2 \cdot 5 \cdot 3^5} \right]$$

$$= \pi \frac{4^3}{3^5} \left(\frac{1}{3} - \frac{1}{5} \right)$$

$$= \pi \frac{4^3}{3^5} \frac{2}{15}$$

4

3



$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

$$\frac{y^2}{4} = 1 - \frac{x^2}{9}$$

$$y^2 = 4 - \frac{4}{9}x^2$$

$$y = \pm\sqrt{4 - \frac{4}{9}x^2}$$

$$\text{Volume} = \pi \int_{-3}^{3} \left(\sqrt{4 - \frac{4}{9}x^2}\right)^2 dx$$

$$= \pi \int_{-3}^{3} 4 - \frac{4}{9}x^2 \, dx$$

$$= \pi \left(4x \Big|_{-3}^{3} - \frac{4}{27}x^3 \Big|_{-3}^{3} \right)$$

$$= \pi \left(24 - \frac{4}{27}(27 + 27) \right)$$

$$= 16\pi$$

5

$$\mathbb{P}(0 \le x \le .5) = \frac{1}{4} \int_0^{.5} \frac{1}{1+\sqrt{x}}\,dx$$

Let $u = 1+\sqrt{x}$, $\sqrt{x} = u-1$

$du = \frac{1}{2\sqrt{x}}dx = \frac{1}{2(u-1)}dx$

i.e. $dx = 2(u-1)du$

<u>Note:</u>

if $x=0$, $u=1$

if $x=.5$ $u = 1+\sqrt{.5}$

$$\mathbb{P}(0 \le x \le .5) = \frac{1}{4} \int_0^{.5} \frac{1}{1+\sqrt{x}}\,dx = \frac{1}{4}\int_1^{1+\sqrt{.5}} \frac{1}{u}\, 2(u-1)\,du$$

$$= \frac{1}{2}\int_1^{1+\sqrt{.5}} 1 - \frac{1}{u}\,du$$

$$= \frac{1}{2}\left(u - \ln|u|\right)\Big|_1^{1+\sqrt{.5}}$$

$$= \frac{1}{2}\left(1+\sqrt{.5} - \ln(1+\sqrt{.5})\right)$$

$$- \frac{1}{2}\left(1 - \cancelto{0}{\ln(1)}\right)$$

$$= \frac{1}{2}\left(\sqrt{.5} - \ln(1+\sqrt{.5})\right)$$

(6) $\int x^2 \sqrt[5]{3x-1}\, dx$ (4)

Let $u = 3x-1$, $x = \frac{u+1}{3}$

$du = 3dx$

Then

$$\int x^2 \sqrt[5]{3x-1}\, dx = \int \left(\frac{u+1}{3}\right)^2 u^{1/5} \frac{du}{3}$$

$$= \frac{1}{27} \int (u^2+2u+1)\, u^{1/5}\, du$$

$$= \frac{1}{27} \int u^{11/5} + 2u^{6/5} + u^{1/5}\, du$$

$$= \frac{1}{27}\cdot\frac{5}{16}(3x-1)^{16/5} + \frac{2}{27}\cdot\frac{5}{11}(3x-1)^{11/5}$$

$$+ \frac{1}{27}\cdot\frac{5}{6}(3x-1)^{6/5} + C$$

⑦ $\int x(\sec^2(2x) - e^{3x})\,dx$

$= \int x\sec^2(2x)\,dx - \int xe^{3x}\,dx$

$u = x \quad dv = \sec^2(2x)dx \qquad u = x \quad dv = e^{3x}dx$

$du = dx \quad v = \frac{\tan(2x)}{2} \qquad du = dx \quad v = \frac{e^{3x}}{3}$

$$= \frac{x\tan(2x)}{2} - \int \frac{\tan(2x)}{2}dx - \left[\frac{xe^{3x}}{3} - \int \frac{e^{3x}}{3}dx\right]$$

$$= \frac{x\tan(2x)}{2} - \frac{1}{4}\ln|\cos(2x)| - \frac{xe^{3x}}{3} + \frac{e^{3x}}{9} + C$$

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⑧ $\int \sec(3x) + \frac{\sin(2x)}{3+\cos(2x)}\,dx$

$= \int \sec(3x)dx + \int \frac{\sin(2x)}{3+\cos(2x)}dx$

$\boxed{\begin{array}{l} u = 3+\cos(2x) \\ du = -2\sin(2x)dx \end{array}}$

$$= \frac{1}{3}\ln|\sec(3x) + \tan(3x)| - \frac{1}{2}\ln|3+\cos(2x)| + C$$