MATH 16B:
TEST 2

SPRING 2007

Name	Key
I.D. Number	

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
Total	70	

(1) a) Sketch the graph of the function

$$f(x) = 2\cos(2x)$$

on the interval $[0, \pi]$.

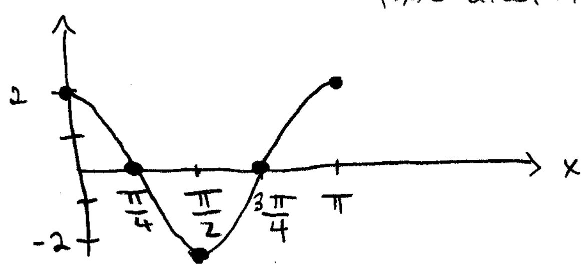
b) Find the area of the region bounded by the x-axis and the graph of the function f on the interval $[0, \pi]$.

c) Find the average value of f on $[0, \pi]$.

d) For what values of x in $[0, \pi]$ does f equal it's average value?

$f(x) = 2\cos(2x)$

a)



b) $$\text{Area} = \int_0^{\pi/4} (2\cos(2x) - 0)\,dx + \int_{\frac{\pi}{4}}^{3\pi/4} (0 - 2\cos(2x))\,dx + \int_{\frac{3\pi}{4}}^{\pi} (2\cos(2x) - 0)\,dx$$

$$= \sin(2x)\Big|_0^{\pi/4} - \sin(2x)\Big|_{\pi/4}^{3\pi/4} + \sin(2x)\Big|_{\frac{3\pi}{4}}^{\pi}$$

$$= \sin(\pi/2) - (\sin(\tfrac{3\pi}{2}) - \sin(\pi/2)) - \sin(\tfrac{3\pi}{2})$$

$$= 1 - (-1 - 1) + 1$$

$$= 4$$

c) $$\text{Average value of } f = \frac{1}{\pi} \int_0^{\pi} 2\cos(2x)\,dx$$

$$= \frac{1}{\pi} \sin(2x)\Big|_0^{\pi} = 0$$

d) $x = \pi/4$ and $x = \frac{3\pi}{4}$.

(2) Suppose that f is an odd function and g is an even function.

a) Is the function $h(x) = f(x)g(x)$ odd, even, or neither? Explain your answer.

This function is odd.

$$h(-x) = f(-x)g(-x) = (-f(x))(g(x)) = -f(x)g(x) = -h(x)$$

b) Is the function $h(x) = (f \circ g)(x) = f(g(x))$ odd, even, or neither? Explain your answer.

$$h(-x) = f(g(-x)) \underset{b/c\ g \text{ is even}}{=} f(g(x)) = h(x)$$

This function is even.

Suppose further that we know

$$\int_0^5 f^2(x)dx = 3 \quad \text{and} \quad \int_{-5}^0 g^2(x)dx = 6.$$

c) Calculate the following integral:

$$\int_{-5}^5 (g(x) - 3\,f(x))^2\,dx.$$

c) $\int_{-5}^{5} (g(x) - 3f(x))^2 dx$

$= \int_{-5}^{5} g^2(x) - 6g(x)f(x) + 9f^2(x)\,dx$

$= 2\int_{-5}^{0} g^2(x)dx - 6\int_{-5}^{5} g(x)f(x)dx + 9 \cdot 2\int_{0}^{5} f^2(x)dx$

$= 12 - 6 \cdot 0 + 18 \cdot 3$

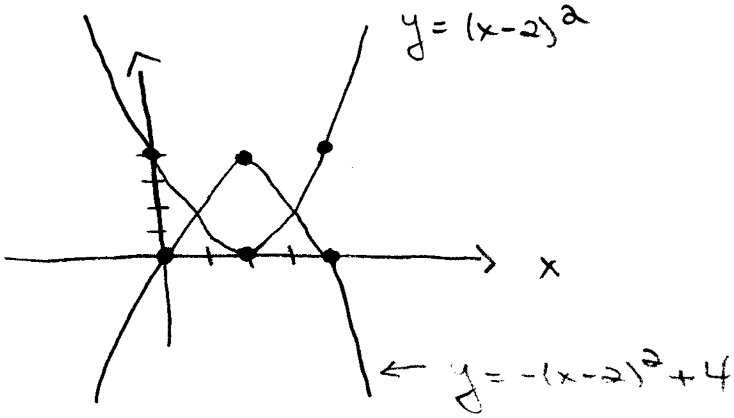
$= 66$

(3) a) Sketch the graphs of $y = (x-2)^2$ and $y = -(x-2)^2 + 4$ on the same axes.

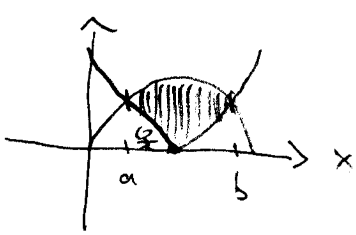
b) Write an expression which calculates the volume of the solid obtained by revolving the region bounded by these curves about the x-axis.

c) Write an expression which calculates the volume of the solid obtained by revolving the region bounded by $y = (x-2)^2$, $y = -(x-2)^2 + 4$, and $x = 0$ about the y-axis.

a)



b)



$$\text{Volume} = \pi\int_a^b \left[-(x-2)^2+4\right]^2 dx - \pi\int_a^b (x-2)^4\,dx$$

a and b are the intersection points:

$$(x-2)^2 = -(x-2)^2 + 4$$

$$\Rightarrow \quad 2(x-2)^2 = 4$$

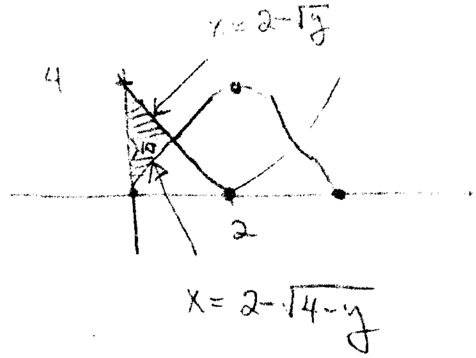
$$(x-2)^2 = 2$$

$$x - 2 = \pm\sqrt{2}$$

$$x = 2 \pm \sqrt{2}$$

So $a = 2 - \sqrt{2}$

and $b = 2 + \sqrt{2}$

c) 

Note:

$y = (x-2)^2$

$\Rightarrow \quad x-2 = \pm\sqrt{y}$

$x = 2 \pm \sqrt{y}$

$y = -(x-2)^2+4$

$(x-2)^2 = 4-y$

$x-2 = \pm\sqrt{4-y}$

$x = 2 \pm \sqrt{4-y}$

Where do they meet?

$2-\sqrt{4-y} = 2-\sqrt{y}$

$\sqrt{y} = \sqrt{4-y}$

$y = 4-y$

$2y = 4$

$y = 2$

$$\text{Volume} = \pi \int_0^2 \left(2-\sqrt{4-y}\right)^2 dy + \pi \int_2^4 \left(2-\sqrt{y}\right)^2 dy$$

(4) a) For any $a > 0$ calculate:

$$\int_0^a \frac{2x}{\sqrt{25-2x^2}}dx.$$

Let $u = 25 - 2x^2$

$du = -4x\,dx$

$\Rightarrow x\,dx = -\frac{1}{4}du$

If $x=0$, then $u = 25 - 2(0)^2$
$u = 25$

If $x = a$, then $u = 25 - 2a^2$

$$\int_0^a \frac{2x}{\sqrt{25-2x^2}}dx = \int_{25}^{25-2a^2} \frac{2(-\frac{1}{4})du}{u^{1/2}}$$

$$= \frac{1}{2}\int_{25-2a^2}^{25} u^{-1/2}du$$

$$= u^{1/2}\Big|_{25-2a^2}^{25}$$

$$= \sqrt{25} - \sqrt{25-2a^2}$$

b) For what value of a is the above integral equal to 1?

We just showed that

$$\int_0^a \frac{2x}{\sqrt{25-2x^2}}dx = \sqrt{25} - \sqrt{25-2a^2}$$

This equals one when

$$\sqrt{25} - \sqrt{25-2a^2} = 1$$

i.e. $4 = \sqrt{25-2a^2}$

$16 = 25 - 2a^2$

$2a^2 = 9$

$a^2 = \frac{9}{2}$ $\quad a = \pm\sqrt{\frac{9}{2}}$

(5) Integrate

$$\int x^2 \sqrt[3]{1-5x}\,dx.$$

Let $u = 1-5x$

$du = -5\,dx$

$\Rightarrow \int x^2 \sqrt[3]{1-5x}\,dx$

$5x = 1-u$

$x = \frac{1-u}{5}$

$$= \int \left(\frac{1-u}{5}\right)^2 \cdot u^{1/3} \left(-\frac{1}{5}\right) du$$

$$= \frac{-1}{5^3} \int u^{1/3}(1-2u+u^2)\,du$$

$$= -\frac{1}{5^3}\left[\frac{u^{4/3}}{4/3} - 2\frac{u^{7/3}}{\frac{7}{3}} + \frac{u^{10/3}}{10/3}\right] + C$$

$$= -\frac{3}{5^3}\left[\frac{1}{4}(1-5x)^{4/3} - \frac{2}{7}(1-5x)^{7/3} + \frac{1}{10}(1-5x)^{10/3}\right] + C$$

(6) Integrate

$$\int 2x \ln[x^4]\,dx.$$

Integrate by Parts

$\int 2x \ln(x^4)\,dx$

Let

$u = \ln(x^4)$	$dv = 2x\,dx$
$du = \frac{1}{x^4} \cdot 4x^3\,dx$	$v = x^2$
$du = \frac{4}{x}\,dx$	

$\int u\,dv = uv - \int v\,du$

$$\int 2x\ln(x^4)\,dx = x^2\ln(x^4) - \int x^2 \cdot \frac{4}{x}\,dx$$

$$= x^2\ln(x^4) - 2x^2 + C$$

(7) Integrate

$$\int \cos\left(\sqrt{x}\right)\, dx$$

Hint: As a first step, try a substitution.

$$\boxed{\begin{aligned} &\text{Let } t = \sqrt{x} \\ &dt = \frac{1}{2\sqrt{x}}\,dx \\ &\Rightarrow dx = 2t\,dt \end{aligned}}$$

$$\int \cos(\sqrt{x})\,dx$$

$$= \int \cos(t)\cdot 2t\,dt$$

Integrate by parts

Let $u = 2t$ $\quad dv = \cos(t)\,dt$

$du = 2dt$ $\quad V = \sin(t)$

$\Rightarrow$

$$\int \cos(\sqrt{x})\,dx = \int 2t\cos(t)\,dt = 2t\sin(t) - 2\int \sin(t)\,dt$$

$$= 2t\sin(t) + 2\cos(t) + C$$

$$= 2\sqrt{x}\,\sin(\sqrt{x}) + 2\cos(\sqrt{x}) + C$$