Key

MATH 16B:
TEST 3

FALL 2006

(1) Integrate the following:

$$\int \frac{x(x+1)(x+2)}{(x-3)(x-4)}\,dx.$$

$$\int \frac{x(x+1)(x+2)}{(x-3)(x-4)}dx = \int \frac{x^3+3x^2+2x}{x^2-7x+12}dx$$ must divide 1st!

$$\begin{array}{r|l} & x \quad +10 \\ \hline x^2-7x+12 & x^3+3x^2+2x \\ & -(x^3-7x^2+12x) \\ \hline & 10x^2-10x \\ & -(10x^2-70x+120) \\ \hline & 60x-120 \end{array}$$

$$\Rightarrow \frac{x^3+3x^2+2x}{x^2-7x+12} = x+10+60\frac{(x-2)}{(x-3)(x-4)}$$

Partial Fractions:

$$\frac{x-2}{(x-3)(x-4)} = \frac{A}{x-3} + \frac{B}{x-4}$$

$$\Rightarrow x-2 = A(x-4) + B(x-3)$$

$$\Rightarrow 1 = A+B \quad \Rightarrow \boxed{A = 1-B = 1-(2) = -1}$$

$$-2 = -4A-3B \quad \Rightarrow -2 = -4(1-B)-3B$$

$$\Rightarrow \boxed{B=2}$$

Thus

$$\int \frac{x(x+1)(x+2)}{(x-3)(x-4)}dx$$

$$= \int x+10+60\cdot\frac{x-2}{(x-3)(x-4)}dx$$

$$= \frac{x^2}{2}+10x+60\left[-\int\frac{1}{x-3}dx + 2\int\frac{1}{x-4}dx\right]$$

$$= \boxed{\frac{x^2}{2}+10x-60\ln|x-3| + 120\ln|x-4| + C}$$

(2) Determine whether the following integrals converge or diverge. If they converge, find their value.

$$a) \qquad \int_0^4 \frac{1}{\sqrt[3]{2x-5}}\,dx,$$

and

$$b) \qquad \int_{-\infty}^{-1} \frac{1}{\sqrt[3]{2x-5}}\,dx.$$

Note: In general,

$$\int \frac{1}{\sqrt[3]{2x-5}}\,dx = \frac{1}{2}\int u^{-1/3}\,du = \frac{1}{2}\,\frac{u^{2/3}}{2/3} + C$$

$$= \frac{3}{4}(2x-5)^{2/3} + C$$

$u = 2x-5$

$du = 2dx$

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a) $$\int_0^4 \frac{1}{\sqrt[3]{2x-5}}\,dx = \int_0^{2.5} \frac{1}{\sqrt[3]{2x-5}}\,dx + \int_{2.5}^4 \frac{1}{\sqrt[3]{2x-5}}\,dx$$

$$= \lim_{a\to 2.5^-} \frac{3}{4}(2x-5)^{2/3}\Big|_0^a + \lim_{a\to 2.5^+} \frac{3}{4}(2x-5)^{2/3}\Big|_a^4$$

Note: $2x-5=0$ when $x = 5/2 = 2.5$

"This bad spot" is in $[0,4]$!

$$= 0 - \frac{3}{4}(-5)^{2/3} + \frac{3}{4}(3)^{2/3} - 0$$

$$= \frac{3}{4}\left[3^{2/3} - 5^{2/3}\right]$$ converges

b) $$\int_{-\infty}^{-1} \frac{1}{\sqrt[3]{2x-5}}\,dx = \lim_{R\to-\infty} \int_R^{-1} \frac{1}{\sqrt[3]{2x-5}}\,dx = \lim_{R\to-\infty} \frac{3}{4}(2x-5)^{2/3}\Big|_R^{-1}$$

$$= \lim_{R\to-\infty} \frac{3}{4}(-7)^{2/3} - \frac{3}{4}(2R-5)^{2/3}$$

$$= -\infty$$ diverges!

(3) Consider a two step experiment. First, toss a six sided die. (As usual, the values on the face of this particular die range from 1 to 6.) If the value of the tossed die is an even number, toss a coin. If the value of the tossed die is an odd number, toss the die again.
a) Write out the sample space of this experiment.

Sample Space	x-value
11	2
12	3
13	4
14	5
15	6
16	7
2H	4
2T	3
31	4
32	5

Sample Space	x value
33	6
34	7
35	8
36	9
4H	6
4T	5
51	6

Sample Space	x-value
52	7
53	8
54	9
55	10
56	11
6H	8
6T	7

b) Define a random variable x which assigns numbers according to the outcomes in the following way:

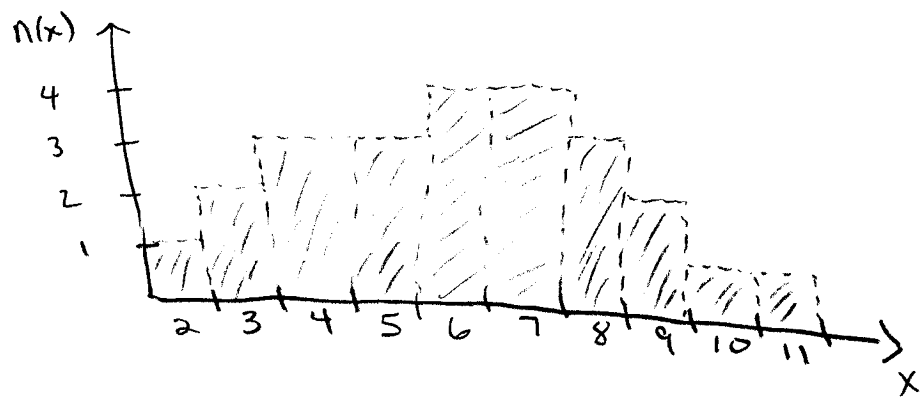
If two die are tossed, then x assigns the value corresponding to the sum of the two face values.

If one die is tossed and then a head occurs, then x assigns the value of the die plus 2.

If one die occurs and then a tail occurs, then x assigns the value of the die plus 1.

Write out a frequency table for x, plot its frequency distribution, and make a table listing the relevant probabilities.

x	n(x)	P(x)
2	1	1/24
3	2	2/24
4	3	3/24
5	3	3/24
6	4	4/24
7	4	4/24
8	3	3/24
9	2	2/24
10	1	1/24
11	1	1/24



(4) Consider a random variable x with frequencies given by

x	n(x)
1	3
2	5
3	2
4	3

$$\mu = E(x) = x_1 P(x_1) + x_2 P(x_2) + x_3 P(x_3) + x_4 P(x_4)$$
$$= 1\cdot\frac{3}{13} + 2\cdot\frac{5}{13} + 3\cdot\frac{2}{13} + 4\cdot\frac{3}{13}$$
$$= \frac{31}{13}$$

a) Find the mean of x.
b) Write an expression which calculates the variance and standard deviaition of x; i.e., plug in the numbers, but you need not find an exact value.

$$V(x) = (x_1-\mu)^2 P(x_1) + (x_2-\mu)^2 P(x_2) + (x_3-\mu)^2 P(x_3) + (x_4-\mu)^2 P(x_4)$$
$$= \left(1-\frac{31}{13}\right)^2\cdot\frac{3}{13} + \left(2-\frac{31}{13}\right)^2\cdot\frac{5}{13} + \left(3-\frac{31}{13}\right)^2\cdot\frac{2}{13} + \left(4-\frac{31}{13}\right)^2\cdot\frac{3}{13}$$

$$\sigma(x) = \sqrt{V(x)} = \sqrt{}$$

(5) Consider the function
$$f(x) = a(x^3 - x + 3).$$
a) For what value of a is f is a probability density function on [-1,1]?
b) For this value of a, find the mean of the continuous random variable x whose density is f.
c) For this value of a, find the variance of x.

a) $$1 = \int_{-1}^{1} f(x)\,dx = a\int_{-1}^{1} x^3\,dx - a\int_{-1}^{1} x\,dx + 3a\int_{-1}^{1} dx$$
$$= 0 - 0 + 3a\cdot 2 \Rightarrow \boxed{a = 1/6}$$

b) $$\mu = E(x) = \int_{-1}^{1} x f(x)\,dx = \frac{1}{6}\left[\int_{-1}^{1} x^4\,dx - \int_{-1}^{1} x^2\,dx + 3\int_{-1}^{1} x\,dx\right]$$
$$= \frac{1}{6}\left[2\int_0^1 x^4\,dx - 2\int_0^1 x^2\,dx + 0\right]$$
$$= \frac{2}{30} - \frac{2}{18} = \boxed{\frac{-2}{45}}$$

c) $$V(x) = \int_{-1}^{1} x^2 f(x)\,dx - \mu^2 = \frac{1}{6}\left[\int_{-1}^{1} x^5\,dx - \int_{-1}^{1} x^3\,dx + 3\int_{-1}^{1} x^2\,dx\right] - \left(\frac{-2}{45}\right)^2$$
$$= \frac{1}{6}\left[0 - 0 + 6\int_0^1 x^2\,dx\right] - \left(\frac{2}{45}\right)^2 = \boxed{\frac{1}{3} - \frac{4}{(45)^2}}$$

(6) Suppose x is a continuous random variable with an exponential probability density function, i.e., $f(x) = ae^{-ax}$ on $[0, \infty)$. If x has median $m = \frac{\ln(2)}{10}$,
a) Find the actual probability density function f.
b) Find the mean of x.
c) Sketch f. Label the mean and median of x indicating which is bigger.

a) $\int_0^m ae^{-ax}\,dx = \frac{1}{2} \Rightarrow -e^{-ax}\Big|_0^m = 1/2 \Rightarrow 1 - e^{-am} = 1/2$

so $e^{-am} = 1/2 \Rightarrow -am = \ln(\frac{1}{2})$

$\Rightarrow a = \frac{-\ln(2)}{-m} = \frac{\ln(2)}{\frac{\ln(2)}{10}} = 10$

b) $\mu = E(x) = \int_0^\infty x f(x)\,dx = \int_0^\infty x \cdot 10e^{-10x}\,dx = \lim_{R\to\infty} \int_0^R 10xe^{-10x}\,dx$

Let $u = 10x$ $\quad dv = e^{-10x}dx$

$du = 10dx$ $\quad v = -\frac{1}{10}e^{-10x}$

$= \lim_{R\to\infty}\left[-xe^{-10x}\Big|_0^R + \int_0^R e^{-10x}\,dx\right] = 0 + \lim_{R\to 0} -\frac{1}{10}e^{-10x}\Big|_0^R$

$= 1/10.$

c)