7.5

4) $f(x,y) = \sqrt{25-(x-2)^2-y^2}$

$f_x = \cancel{\tfrac{1}{2}}(25-(x-2)^2-y^2)^{-1/2}(-x+2)(\cancel{2})$

$f_y = \tfrac{1}{\cancel{2}}(25-(x-2)^2-y^2)^{-1/2}(-\cancel{2}y)$

- using product rule

$f_{xx} = (25-(x-2)^2-y^2)^{-1/2}(-1) + (-\tfrac{1}{2})(25-(x-2)^2-y^2)^{-3/2}(-x+2)(2)(-x+2)$

$= \dfrac{y^2+(x-2)^2-25 \quad -(x-2)^2}{(25-(x-2)^2-y^2)^{3/2}} = \dfrac{y^2-25}{(25-(x-2)^2-y^2)^{3/2}}$

$f_{yy} = (25-(x-2)^2-y^2)^{-1/2}(-1) + (-\tfrac{1}{2})(25-(x-2)^2-y^2)^{-3/2}(-2y)(-y)$

$= \dfrac{y^2+(x-2)^2-25 \quad -y^2}{(25-(x-2)^2-y^2)^{3/2}} = \dfrac{(x-2)^2-25}{(25-(x-2)^2-y^2)^{3/2}}$

$f_{xy} = (25-(x-2)^2-y^2)^{-1/2}(0) + (-\tfrac{1}{2})(25-(x-2)^2-y^2)^{-3/2}(-x+2)(2)(-y)$

$= \dfrac{(-x+2)(y)}{(25-(x-2)^2-y^2)^{3/2}}$

- find critical points

$f_x = \dfrac{-x+2}{\sqrt{25-(x-2)^2-y^2}} = 0 \implies x=2$

$f_y = \dfrac{-y}{\sqrt{25-(x-2)^2-y^2}} = 0 \implies y=0$

$\left.\begin{array}{l} f_{xx}(2,0) = -\tfrac{1}{5} \\ f_{yy}(2,0) = -\tfrac{1}{5} \\ f_{xy}(2,0) = 0 \end{array}\right\} \implies d = f_{xx}(2,0)\cdot f_{yy}(2,0) - [f_{xy}(2,0)]^2 = \tfrac{1}{25} > 0$

$(2,0,f(2,0)) \rightarrow$ <u>(2,0,5) is a relative max.</u>

since $d>0$ and $f_{xx}<0$

6) $f(x,y) = 9 - (x-3)^2 - (y+2)^2$

- hold 'y' constant

$f_x = -2(x-3)$

- hold 'x' constant

$f_y = -2(y+2)$

$$\left.\begin{aligned} f_{xx} &= -2 \\ f_{yy} &= -2 \\ f_{xy} &= 0 \end{aligned}\right\} \quad d = 4 > 0$$

- find critical points

$-2x + 6 = 0 \Rightarrow x = 3$

$-2x - 4 = 0 \Rightarrow y = -2$

$f(3,-2) = 9$

$\Rightarrow$ $\underline{(3,-2,9) \text{ is a rel. max.}}$

since $d > 0$ and $f_{xx} < 0$

11) $f(x,y) = 3x^2 + 2y^2 - 12x - 4y + 7$

$$\left.\begin{aligned} f_x &= 6x - 12 \\ f_y &= 4y - 4 \end{aligned}\right\} \Rightarrow \begin{aligned} 6x - 12 &= 0 \quad & x = 2 \\ 4y - 4 &= 0 \quad & y = 1 \end{aligned}$$

$$\left.\begin{aligned} f_{xx} &= 6 \\ f_{yy} &= 4 \\ f_{xy} &= 0 \end{aligned}\right\} \quad d = 24 > 0$$

$f(2,1) = -7$

$\Rightarrow$ $\underline{(2,1,-7) \text{ is a rel. min.}}$

since $d > 0$ and $f_{xx} > 0$

16) $f(x,y) = 12x + 12y - xy - x^2 - y^2$

$$\left.\begin{aligned} f_x &= 12 - y - 2x \\ f_y &= 12 - x - 2y \end{aligned}\right\} \Rightarrow$$ remember they have to simultaneously equal zero

$12 - y - 2x = 0 \Rightarrow y = 12 - 2x$

$12 - x - 2y = 0 \Rightarrow 12 - x - 2(12 - 2x) = 0 \Rightarrow -12 + 3x = 0 \Rightarrow \underline{x = 4}$

$y = 12 - 2(4) = 4 \Rightarrow \underline{y = 4}$

$$\left.\begin{aligned} f_{xx} &= -2 \\ f_{yy} &= -2 \\ f_{xy} &= -1 \end{aligned}\right\} \Rightarrow d = 4 - 1 = 3 > 0$$

$f(4,4) = 48$

$\Rightarrow$ $\underline{(4,4,48) \text{ is a rel. max}}$ since $d > 0$ and $f_{xx} < 0$

17) $f(x,y) = (x^2 + 4y^2)e^{(1-x^2-y^2)}$

$f_x = 2xe^{(1-x^2-y^2)} + (x^2+4y^2)(-2x)e^{(1-x^2-y^2)} = 2xe^{(1-x^2-y^2)}(1 - x^2 - 4y^2)$

$f_y = 8ye^{(1-x^2-y^2)} + (x^2+4y^2)(-2y)e^{(1-x^2-y^2)} = 2ye^{(1-x^2-y^2)}(4 - x^2 - 4y^2)$

recall $\underline{e^x \neq 0}$ for any $x$ only $\lim_{x\to-\infty} e^x = 0$

(1) $2x(1 - x^2 - 4y^2) = 0$

(2) $2y(4 - x^2 - 4y^2) = 0$

let $x = 0$ then equation (1) is always zero and equation (2) equals zero when $y = 0$ or $4 - 4y^2 = 0$

$y = \pm 1$

let $y = 0$ equation (2) is always zero and equation (1) equals zero when $x = 0$ or $1 - x^2 = 0$

$x = \pm 1$

points so far

(0,0)
(0,1)
(0,-1)
(1,0)
(-1,0)

finally, we need to check for points when

$$1 - x^2 - 4y^2 = 0$$

and $4 - x^2 - 4y^2 = 0$ simultaneously

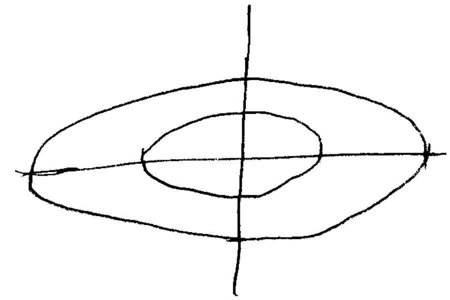
$$\Downarrow$$

$$x^2 + \frac{y^2}{(1/2)^2} = 1$$

and

$$\frac{x^2}{(2)^2} + y^2 = 1$$

these are equations of ellipses $\Rightarrow$



since they don't have any points that meet there are no more critical points.

$$f_{xx} = 2e^{(1-x^2-y^2)}(1 - 5x^2 + 2x^4 - 4y^2 + 8x^2y^2)$$

$$f_{yy} = 2e^{(1-x^2-y^2)}(4 - x^2 - 2y^2 + 8y^4 + 2x^2y^2)$$

$$f_{xy} = -4xye^{(1-x^2-y^2)}(5 - x^2 - 4y^2)$$

$d(0,0) > 0$ and $f_{xx}(0,0) > 0 \Rightarrow (0,0,0)$ is a rel min.

$d(0,1) > 0$ and $f_{xx}(0,1) < 0$

$d(0,-1) > 0$ and $f_{xx}(0,-1) < 0$ $\Bigg\} \Rightarrow$ $(0,1,4)$, $(0,-1,4)$ are rel. max.

$d(1,0) < 0$

$d(-1,0) < 0$ $\Rightarrow$ $(1,0,1)$, $(-1,0,1)$ are saddle points

22) $d = f_{xx} \cdot f_{yy} - [f_{xy}]^2 = 24 - 9 = 15 > 0$

$f_{xx} < 0 \Rightarrow$ rel. max.

28) $f(x,y) = x^3 + y^3 - 3x^2 + 6y^2 + 3x + 12y + 7$

$f_x = 3x^2 - 6x + 3 = 3(x-1)^2$ $\Rightarrow$ critical point $(1, -2)$

$f_y = 3y^2 + 12y + 12 = 3(y+2)^2$

$$\left.\begin{array}{l} f_{xx} = 6x - 6 \\ f_{yy} = 6y + 12 \\ f_{xy} = 0 \end{array}\right\} \Rightarrow d(0,0) = 0$$

thus the test <u>fails</u> for $(1,-2)$

By testing nearby points we find we have a saddle point.

30) $f(x,y) = (x^2+y^2)^{2/3}$

$f_x = \dfrac{4x}{3(x^2+y^2)^{1/3}} = 0$ for $x=0$ $\Rightarrow$ crit. pt. $(0,0)$

$f_y = \dfrac{4y}{3(x^2+y^2)^{1/3}} = 0$ for $y=0$

$$\left.\begin{array}{l} f_{xx} = \dfrac{4}{9}\dfrac{x^2+3y^2}{(x^2+y^2)^{4/3}} \\ f_{yy} = \dfrac{4}{9}\dfrac{y^2+3x^2}{(x^2+y^2)^{4/3}} \\ f_{xy} = -\dfrac{8}{9}\dfrac{xy}{(x^2+y^2)^{4/3}} \end{array}\right\}$$

$d(0,0) = 0$ thus the test <u>fails</u> for $(0,0)$

But all points around $(0,0)$ are positives so $(0,0,0)$ is a rel. min.

34) maximize $P = xy^2z$

constraint $x+y+z = 32 \Rightarrow z = 32 - x - y$

$P = xy^2(32-x-y)$

$P_x = 32y^2 - 2xy^2 - y^3 = y^2(32-2x-y) = 0$

$P_y = 64xy - 2x^2y - 3xy^2 = xy(64-2x-3y) = 0$

although $x=0$ and $y=0$ satisfy the equation, those values don't maximize $P$ thus we ignore them

$\Downarrow$

$$\left.\begin{array}{l} 32-2x-y = 0 \\ 64-2x-3y = 0 \end{array}\right\} \Rightarrow \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 32 \\ 64 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3/4 & -1/4 \\ -1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 32 \\ 64 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 24-16 \\ -16+32 \end{bmatrix} = \begin{bmatrix} 8 \\ 16 \end{bmatrix}$$

This is a good use of linear algebra but feel free to use substitution!

$z = 32 - 8 - 16 = 8$ $\quad P = (8)(16)^2(8) = 16{,}384$ for $\boxed{\begin{array}{l} x = 8 \\ y = 16 \\ z = 8 \end{array}}$

38) $R = 500p_1 + 800p_2 + 1.5p_1p_2 - 1.5p_1^2 - p_2^2$

$R_{p_1} = 500 + 1.5p_2 - 3p_1 = 0$ $\quad p_1 = \frac{500 + 1.5p_2}{3}$

$R_{p_2} = 800 + 1.5p_1 - 2p_2 = 0$ $\quad 800 + \frac{1}{2}(500 + 1.5p_2) - 2p_2 = 0$

$\Rightarrow 1050 - 1.25p_2 = 0$

$\Rightarrow p_2 = \$840$

$p_1 = \frac{500 + 1.5(840)}{3} \approx \$586.67$

$R_{p_1p_1} = -3$

$R_{p_2p_2} = -2$

$R_{p_1p_2} = 1.5$

$d = 6 - (1.5)^2 > 0$ and $R_{p_1p_1} < 0$

$\Rightarrow$ $p_1 = \$586.67$, $p_2 = \$840.00$ is a relative max

43) $x$ = length, $y$ = width, $z$ = height

girth $= 2y + 2z$

constraint $\Rightarrow x + (2y + 2z) = 144$

maximize volume $\Rightarrow V = xyz$

$V = yz(144 - 2y - 2z)$ ignoring $z = 0$ and $y = 0$

$V_y = 144z - 4yz - 2z^2 = 2z(72 - 2y - z)$

$V_z = 144y - 2y^2 - 4yz = 2y(72 - y - 2z)$

$\Rightarrow$ $72 - 2y - z = 0$, $72 - y - 2z = 0$

$z = 72 - 2y$

$\Rightarrow 72 - y - 2(72 - 2y) = 0 \Rightarrow -72 + 3y = 0$

$\Rightarrow \boxed{y = 24}$

$z = 72 - 2(24) \Rightarrow \boxed{z = 24}$

$x = 144 - 2(24) - 2(24) \Rightarrow \boxed{x = 48}$

7.6

4) $F(x,y,\lambda) = x^2+y^2 - \lambda(-2x-4y+5)$

$F_x = 2x + 2\lambda = 0 \Rightarrow x = -\lambda$

$F_y = 2y + 4\lambda = 0 \Rightarrow y = -2\lambda$

$F_\lambda = 2x + 4x - 5 = 0 \Rightarrow -2\lambda - 8\lambda = 5 \Rightarrow \lambda = \frac{-1}{2}$

$\Rightarrow \boxed{\begin{matrix} x = \frac{1}{2} \\ y = 1 \end{matrix}}$ $\quad f(\frac{1}{2}, 1) = \frac{5}{4}$

7) $F(x,y,\lambda) = 3x + xy + 3y - \lambda(x+y-25)$

$F_x = 3 + y - \lambda = 0 \quad \lambda = 3+y$

$F_y = x + 3 - \lambda = 0 \quad \lambda = 3+x$ $\Rightarrow x = y$

$F_\lambda = 25 - x - y = 0 \Rightarrow 25 - 2x = 0 \Rightarrow \boxed{\begin{matrix} x = \frac{25}{2} \\ y = \frac{25}{2} \end{matrix}}$

$f(12.5, 12.5) = 231.25$

10) $F(x,y,\lambda) = (x^2+y^2)^{\frac{1}{2}} - \lambda(2x+4y-15)$

$F_x = x(x^2+y^2)^{-\frac{1}{2}} - 2\lambda = 0 \quad \lambda = \frac{x}{2\sqrt{x^2+y^2}}$

$F_y = y(x^2+y^2)^{-\frac{1}{2}} - 4\lambda = 0 \quad \lambda = \frac{y}{4\sqrt{x^2+y^2}}$ $\Rightarrow \frac{x}{2} = \frac{y}{4} \Rightarrow y = 2x$

$F_\lambda = -2x - 4y + 15 = 0 \Rightarrow -2x - 4(2x) + 15 = 0 \Rightarrow \boxed{\begin{matrix} x = \frac{3}{2} \\ y = 3 \end{matrix}}$

$f(\frac{3}{2}, 3) = \frac{3\sqrt{5}}{2}$

14) $F(x,y,z,\lambda) = xyz - \lambda(x+y+z-6)$

$F_x = yz - \lambda = 0 \quad \lambda = yz$

$F_y = xz - \lambda = 0 \quad \lambda = xz$ $\Rightarrow x = y$ $\Rightarrow x = y = z$

$F_z = xy - \lambda = 0 \quad \lambda = xy$

$F_\lambda = -x - y - z + 6 = 0 \Rightarrow -3x + 6 = 0 \Rightarrow \boxed{\begin{matrix} x = 2 \\ y = 2 \\ z = 2 \end{matrix}}$

$f(2,2,2) = 8$

17) $F(x,y,z,\lambda) = x+y+z - \lambda(x^2+y^2+z^2-1)$

$F_x = 1 - 2\lambda x = 0 \quad \lambda = \frac{1}{2x}$

$F_y = 1 - 2\lambda y = 0 \quad \lambda = \frac{1}{2y}$

$F_z = 1 - 2\lambda z = 0 \quad \lambda = \frac{1}{2z}$

$F_\lambda = -x^2 - y^2 - z^2 + 1 = 0$

$\left.\begin{matrix}\lambda = \frac{1}{2x} \\ \lambda = \frac{1}{2y}\end{matrix}\right\} \Rightarrow x = y \; \Big\} \; x = y = z$

$\Rightarrow 3x^2 = 1 \Rightarrow \boxed{\begin{matrix} x = \sqrt{1/3} = \frac{1}{\sqrt{3}} \\ y = 1/\sqrt{3} \\ z = 1/\sqrt{3} \end{matrix}}$

$f(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}) = \sqrt{3}$