(1) Implicitly differentiate

$$x^2 + y^2 = cy$$

$\Rightarrow$

$$2x + 2yy' = cy'$$

or

$$(2y - c)y' = -2x$$

Thus

$$y' = \frac{-2x}{2y-c}$$

$\boxed{\text{mult. by } \frac{y}{y}}$

$$\longrightarrow y' = \frac{-2xy}{(2y-c)y} = \frac{-2xy}{2y^2 - cy}$$

$\boxed{\text{Use } cy = x^2 + y^2} \longrightarrow$

$$= \frac{-2xy}{2y^2 - (x^2+y^2)}$$

$$= \frac{-2xy}{y^2 - x^2}$$

$$= \frac{2xy}{x^2 - y^2} \quad \checkmark$$

(2) This is separable.

$$\frac{dy}{dx} = \frac{y\cos(x)}{1+2y^2}$$

$$\Rightarrow \quad \frac{1+2y^2}{y}\frac{dy}{dx} = \cos(x)$$

$$\Rightarrow \quad \int \left(\frac{1}{y} + 2y\right)dy = \int \cos(x)\,dx$$

$$\boxed{\ln|y| + y^2 = \sin(x) + C}$$

This is the general solution.

The particular solution that passes through $(0,1)$ satisfies

$$\ln(1) + (1)^2 = \sin(0) + C$$

i.e. $C = 1$

Thus

$$\boxed{\ln|y| + y^2 = \sin(x) + 1}$$

③ This eqn. is 1^{st} order linear.

$$(1+x^2)y' + 4xy = \frac{\sqrt{x}}{(1+x^2)} \qquad \text{for } x>0$$

$$y' + \frac{4x}{1+x^2}y' = \frac{\sqrt{x}}{(1+x^2)^2}$$

Here $P(x) = \frac{4x}{1+x^2}$ and $Q(x) = \frac{\sqrt{x}}{(1+x^2)^2}$

Thus

$$u(x) = e^{\int P(x)dx} = e^{\int \frac{4x}{1+x^2}dx} = e^{2\int \frac{1}{t}dt} = e^{2\ln|t|}$$

Let $t = 1+x^2$

$dt = 2x\,dx$

$$= e^{\ln(t^2)}$$

$$= t^2$$

and so

$$= (1+x^2)^2$$

$$y(x) = \frac{1}{u(x)}\int Q(x)u(x)\,dx = \frac{1}{(1+x^2)^2}\int \frac{\sqrt{x}}{\cancel{(1+x^2)^2}}\cdot\cancel{(1+x^2)^2}\,dx$$

$$= \frac{1}{(1+x^2)^2}\left[\frac{2}{3}x^{3/2} + C\right]$$

④

$$\frac{dA}{dt} = rA + P$$

1^{st} order linear.

$\boxed{\text{Not in standard form!}}$

$$\frac{dA}{dt} - rA = P$$

$$u(t) = e^{-\int r\,dt} = e^{-rt}$$

$$A(t) = \frac{1}{u(t)} \int P u(t)\,dt = e^{rt} \int P e^{-rt}\,dt$$

$$= e^{rt}\left[-\frac{P}{r}e^{-rt} + C\right]$$

$$= -\frac{P}{r} + C e^{rt}$$

Since $0 = A(0) = -\frac{P}{r} + C$

$C = P/r$ so that

$$\boxed{A(t) = \frac{P}{r}\left(-1 + e^{rt}\right)}$$

(5) Find the center and radius of the sphere given by (3)

$$2x^2 + 2y^2 + 2z^2 - 4x - 12y - 8z + 3 = 0$$

Dividing 2 and rearrange

$$x^2 - 2x + y^2 - 6y + z^2 - 4z = -3/2$$

$$x^2 - 2x + 1 + y^2 - 6y + 9 + z^2 - 4z + 4 = -3/2 + 1 + 9 + 4$$

$$(x-1)^2 + (y-3)^2 + (z-2)^2 = \frac{25}{2}$$

$\boxed{C = (1, 3, 2)}$ $\boxed{r = \frac{5}{\sqrt{2}}}$