Key to take test 2B

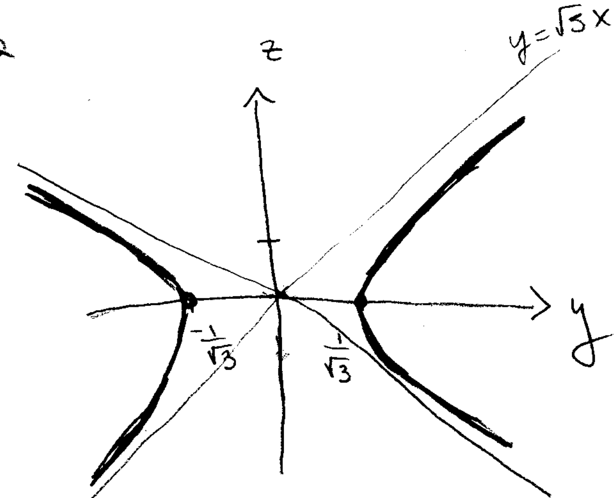
1) $6x^2 - 9y^2 + 3z^2 = 3$

a) if $x = -1$, then $\Rightarrow$ $3 = 6 - 9y^2 + 3z^2$

or $\underline{\underline{x = 1}}$

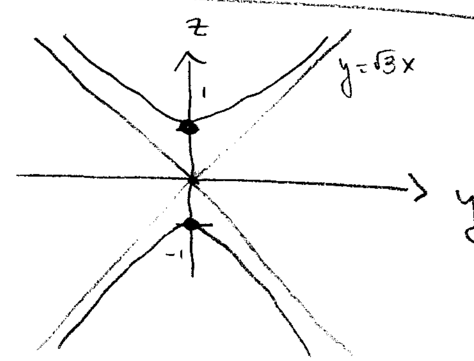
$9y^2 - 3z^2 = 3$

$\frac{y^2}{\frac{1}{3}} - z^2 = 1$



if $x = 0$, then $\Rightarrow$ $3z^2 - 9y^2 = 3$

$z^2 - \frac{y^2}{\frac{1}{3}} = 1$

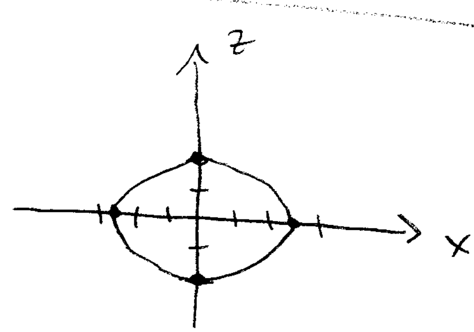


b) if $y = -1$ or $y = +1$, then

$6x^2 - 9 + 3z^2 = 3$

$6x^2 + 3z^2 = 12$

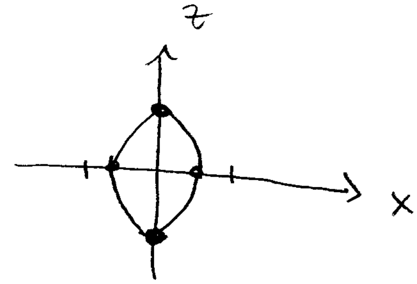
$\frac{x^2}{6} + \frac{z^2}{4} = 1$



if $y = 0$, then

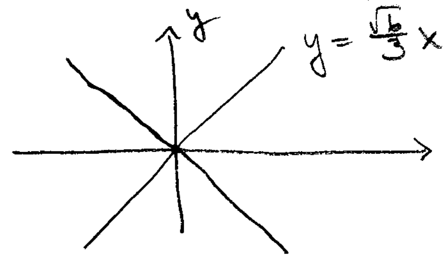
$6x^2 + 3z^2 = 3$

$\frac{x^2}{\frac{1}{2}} + z^2 = 1$



i) if $z=-1$ or $z=+1$, then $6x^2 - 9y^2 + 3 = 3$

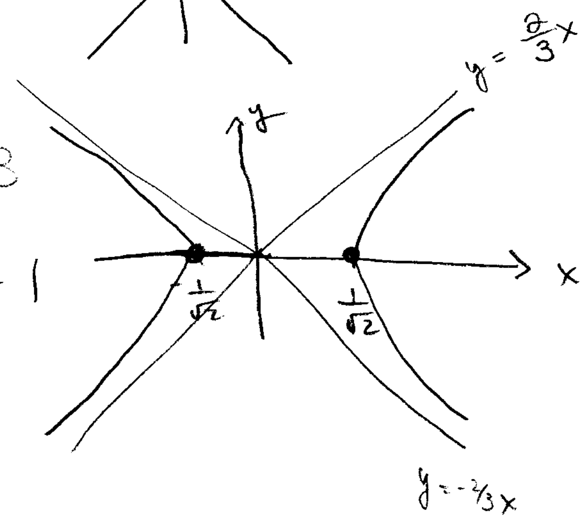
$$9y^2 = 6x^2 \Rightarrow y = \pm \frac{\sqrt{6}}{3} x$$



if $z=0$, then

$$6x^2 - 9y^2 = 3$$

$$\frac{x^2}{\frac{1}{2}} - \frac{y^2}{\frac{1}{3}} = 1$$



---

j) $f(x,y) = \dfrac{x^2}{\sqrt{y}} - e^{\sqrt[3]{x} - y^2} + \cos(2\sqrt{xy})$

$$f_x(x,y) = \frac{2x}{\sqrt{y}} - e^{\sqrt[3]{x} - y^2} \cdot \frac{1}{3} x^{-2/3} - \sin(2\sqrt{x}\sqrt{y}) \frac{\sqrt{y}}{\sqrt{x}}$$

$$f_y(x,y) = -\frac{1}{2} x^2 y^{-3/2} - e^{\sqrt[3]{x} - y^2} (-2y) - \sin(2\sqrt{x}\sqrt{y}) \frac{\sqrt{x}}{\sqrt{y}}$$

3 $f(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$

$f_x(x,y) = 6xy - 6x = 6x(y-1)$
$f_y(x,y) = 3x^2 + 3y^2 - 6y$

The partial derivatives always exist, so only critical points come from $f_x = f_y = 0$.

$f_x = 0 \Rightarrow x = 0$ or $y = 1$

$f_y = 0 \Rightarrow 3x^2 + 3y^2 - 6y = 0$ if $x = 0 \to y(y-2) = 0$
ie $y = 0$ or $y = 2$

if $y = 1$, then $3x^2 + 3 - 6 = 0$
i.e. $x = \pm 1$

Thus there are 4 critical points: $P_1 = (0,0)$, $P_2 = (0,2)$, $P_3 = (1,1)$, $P_4 = (-1,1)$

Note:

$f_{xx}(x,y) = 6(y-1)$
$f_{xy} = 6x$
$f_{yy} = 6y - 6 = 6(y-1)$

Thus

$d(x,y) = f_{xx}(x,y)\, f_{yy}(x,y) - (f_{xy}(x,y))^2$
$= 36(y-1)^2 - 36x^2$

For $P_1 = (0,0)$,

$d(0,0) = 36 > 0$ and $f_{xx}(0,0) = -6$

$\Rightarrow P_1$ is a relative maximum

For $P_2 = (0,2)$,

$d(0,2) = 36 > 0$ and $f_{xx}(0,2) = 6$

$\Rightarrow P_2$ is a relative minimum.

For $P_3 = (1,1)$

$d(1,1) = -36 < 0 \Rightarrow P_3$ is a saddle point

For $P_4 = (-1,1)$

$d(-1,1) = -36 < 0 \Rightarrow P_4$ is a saddle point

---

4) Minimize $f(x,y,z) = x^3+y^3+z^3$

subject to $x^2+y^2+z^2 = 12$.

$F(x,y,z,\lambda) = x^3+y^3+z^3 - \lambda(x^2+y^2+z^2-12)$

$F_x(x,y,z,\lambda) = 3x^2 - 2\lambda x$ $\quad$ $F_x = 0 \Rightarrow x(3x-2\lambda) = 0$ $\quad$ $x = \frac{2\lambda}{3}$

$F_y(x,y,z,\lambda) = 3y^2 - 2\lambda y$ $\quad$ $F_y = 0 \Rightarrow y(3y-2\lambda) = 0$ $\quad$ $y = \frac{2\lambda}{3}$

$F_z(x,y,z,\lambda) = 3z^2 - 2\lambda z$ $\quad$ $F_z = 0 \Rightarrow z(3z-2\lambda) = 0$ $\quad$ $z = \frac{2\lambda}{3}$

$F_\lambda(x,y,z,\lambda) = -(x^2+y^2+z^2-12)$

④ (continued)

$F_\lambda = 0 \quad \Rightarrow \quad x^2 + y^2 + z^2 = 12$

$$\frac{4\lambda^2}{9} + \frac{4\lambda^2}{9} + \frac{4\lambda^2}{9} = 12$$

$$\frac{4\lambda^2}{3} = 12$$

$$\lambda^2 = 9$$

$$\lambda = \pm 3$$

as we want $x \geq 0$, $y \geq 0$ and $z \geq 0$ choose $\lambda = 3$

$x = 2, \; y = 2, \; z = 2.$

and

$f(2,2,2) = 8 + 8 + 8 = 24$

---

5) $\displaystyle\int_0^1 \int_{1-x}^{1+x} (2x - 3y^2)\, dy\, dx$

$$= \int_0^1 (2xy - y^3) \Big|_{1-x}^{1+x} dx$$

$$= \int_0^1 \left[2x(1+x) - (1+x)^3\right] - \left[2x(1-x) - (1-x)^3\right] dx$$

$$= \int_0^1 4x^2 - (1+x)^3 + (1-x)^3\, dx$$

$$= \frac{4x^3}{3} - \frac{(1+x)^4}{4} - \frac{(1-x)^4}{4} \Big|_0^1$$

$$= \left(\frac{4}{3} - \frac{2^4}{2^2}\right) - \left(-\frac{1}{4} - \frac{1}{4}\right) = -\frac{13}{6}$$