# Key to Fake Test 3A



1. Consider

$$a_n = \frac{4 - 3n + 5n^2}{-3n^2 + 2n - 1}$$

a)

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{5n^2 - 3n + 4}{-3n^2 + 2n - 1}$$

$$= \lim_{n\to\infty} \frac{n^2\left(5 - \frac{3}{n} + \frac{4}{n^2}\right)}{n^2\left(-3 + \frac{2}{n} - \frac{1}{n^2}\right)}$$

$$= -\frac{5}{3}$$

b) Since $\lim_{n\to\infty} a_n = -5/3 \neq 0$, the series

$\sum_{n=0}^{\infty} a_n$ diverges by the n-th term test.

(2) Consider

$$\sum_{n=2}^{\infty} \left(\frac{2}{3}\right)^n - \left(\frac{3}{4}\right)^n$$

a) Does this series converge?

Note: $\sum_{n=2}^{\infty} \left(\frac{2}{3}\right)^n$ is a geometric series with $r = 2/3 < 1$ so it converges.

$\sum_{n=2}^{\infty} \left(\frac{3}{4}\right)^n$ is a geometric series with $r = 3/4 < 1$ so it converges.

In this case the difference

$\sum_{n=2}^{\infty} \left(\frac{2}{3}\right)^n - \left(\frac{3}{4}\right)^n$ also converges.

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b) Recall: $\sum_{n=0}^{\infty} a r^n = \frac{a}{1-r}$ if $r < 1$

So $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \frac{1}{1-\frac{2}{3}} = 3$

and $\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n = \frac{1}{1-3/4} = 4$

Unfortunately these sums begin at $n=2$ not $n=0$!

I have subtracted the 1st two terms in both series!

So

$$\sum_{n=2}^{\infty} \left(\tfrac{2}{3}\right)^n - \left(\tfrac{3}{4}\right)^n = \left[\sum_{n=0}^{\infty} \left(\tfrac{2}{3}\right)^n - \left(1+\tfrac{2}{3}\right)\right] - \left[\sum_{n=0}^{\infty} \left(\tfrac{3}{4}\right) - \left(1+\tfrac{3}{4}\right)\right]$$

$$= \left[3 - 1 - \tfrac{2}{3}\right] - \left[4 - 1 - \tfrac{3}{4}\right]$$

$$= 1 + \tfrac{1}{3} - 2 - \tfrac{1}{4}$$

$$= -1 + \tfrac{1}{12}$$

$$= -\tfrac{11}{12}$$

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3

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\pi/2}}$ converges by p-series since $p = \pi/2 > 1$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{3n+1}$

<u>Ratio test</u>. Here $a_n = \dfrac{5^n}{3n+1}$

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{5^{n+1}}{3(n+1)+1}}{\dfrac{5^n}{3n+1}} = \frac{5^{n+1}}{5^n} \cdot \frac{3n+1}{3n+4} = 5\,\frac{3n+1}{3n+4}$$

Thus

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} 5\left|\frac{3n+1}{3n+4}\right|$$

$$= \lim_{n\to\infty} 5\left|\frac{n(3+\frac{1}{n})}{n(3+\frac{4}{n})}\right|$$

$$= 5 > 1$$

So the series in b) diverges by the ratio Test.

(4) Find the Taylor series for $f(x) = x^2+3x-1$ at $a = 2$.

$f(x) = x^2+3x-1$

$f'(x) = 2x+3$

$f''(x) = 2$

[illegible]

$f^n(x) = 0 \quad n \geq 3$.

The Taylor series is then

$$f(x) = \sum_{n=0}^{\infty} a_n (x-2)^n$$

where

$$a_n = \frac{f^n(2)}{n!}$$

Here

$$a_0 = \frac{f(2)}{1} = \frac{4+6-1}{1} = 9$$

$$a_1 = \frac{f'(2)}{1} = \frac{4+3}{1} = 7$$

$$a_2 = \frac{f''(2)}{2!} = \frac{2}{2} = 1$$

$$a_3 = 0$$

So

$$\boxed{x^2+3x-1 = 9 + 7(x-2) + 1(x-2)^2 + 0}$$

(5) Find the $4^{\text{th}}$ degree Taylor polynomial  (3)

for

$$f(x) = \frac{1}{(1-x)^2} \qquad \text{centered at } a=0.$$

$$f(x) = \frac{1}{(1-x)^2} = (1-x)^{-2}$$

$$f'(x) = -2(1-x)^{-2-1}(-1)$$

$$f''(x) = (-2)(-3)(1-x)^{-2-2}(-1)^2$$

$$f'''(x) = (-2)(-3)(-4)(1-x)^{-2-3}(-1)^3$$

$$f^{iv}(x) = (-2)(-3)(-4)(-5)(1-x)^{-2-4}(-1)^4$$

$\Rightarrow$ $f(0) = 1$, $f'(0) = 2$, $f''(0) = 6$, $f'''(0) = 24$

$f^{iv}(0) = 20$

Thus

$$\frac{1}{1-x^2} = f(x) \approx S_4(x) = \sum_{n=0}^{4} \frac{f^{(n)}(0)}{n!} x^n$$

$$= 1 + 2x + 3x^2 + 4x^3 + 5x^4$$