Key to Fake Test 3B

1) If a sequence is arithmetic then

$a_n = an + b$ where a and b are constants.

If $a_0 = 2$ Then

$2 = a(0) + b \Rightarrow b = 2.$

If $a_1 = 0$, then $0 = a(1) + 2 \Rightarrow a = -2$

Check

$a_n = -2n + 2$ ✓

Does the Series

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} -2n+2$$

Converge? No, The terms do not go to zero

So the series diverges by the n-th term test.

(2) Consider

$$\sum_{n=0}^{\infty} 2\,\frac{(-1)^n}{5^n}$$

a) Does the series converge?

Yes, this series is geometric with $a=2$ and $r=\frac{-1}{5}$. (Note: It converges because $|r|<1$.)

b) $$\sum_{n=0}^{\infty} 2\,\frac{(-1)^n}{5^n} = \frac{2}{1-\left(\frac{-1}{5}\right)} = \frac{2}{\frac{6}{5}} = \frac{5}{3}$$

c) $$\sum_{n=0}^{N} 2\left(\frac{-1}{5}\right)^n = \frac{2\left(1-\left(\frac{-1}{5}\right)^{N+1}\right)}{1-\left(\frac{-1}{5}\right)} = \frac{5}{3}\left(1-\left(\frac{-1}{5}\right)^{N+1}\right)$$

3) Do the series converge?

a) $$\sum_{n=1}^{\infty} \frac{(2n)!}{(3n)!}$$

Use Ratio Test. Here $a_n = \frac{(2n)!}{(3n)!}$

So

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(2(n+1))!}{(3(n+1))!}}{\frac{(2n)!}{(3n)!}} = \frac{(2n+2)!}{(2n)!} \frac{(3n)!}{(3n+3)!} = \frac{(2n+2)(2n+1)}{(3n+3)(3n+2)(3n+1)}$$

Thus

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 0$$ as the denominator has a larger degree

In this case, the series converges,

b) $$\sum_{n=1}^{\infty} \frac{\sqrt[5]{n}}{\sqrt[3]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{3}-\frac{1}{5}}}$$

This is a p-series with

$p = \frac{1}{3} - \frac{1}{5} = \frac{2}{15} < 1$, so the series diverges.

(4) $f(x) = \sqrt[4]{1+x} = (1+x)^{1/4}$ Find the Maclaurin series

$f'(x) = \frac{1}{4}(1+x)^{1/4 - 1}$

$f''(x) = \frac{1}{4}\left(\frac{1}{4}-1\right)(1+x)^{\frac{1}{4}-2}$

$f'''(x) = \frac{1}{4}\left(\frac{1}{4}-1\right)\left(\frac{1}{4}-2\right)(1+x)^{\frac{1}{4}-3}$

$f^{4}(x) = \frac{1}{4}\left(\frac{1}{4}-1\right)\left(\frac{1}{4}-2\right)\left(\frac{1}{4}-3\right)(1+x)^{\frac{1}{4}-4}$

$f^{n}(x) = \frac{1}{4}\left(\frac{1}{4}-1\right)\left(\frac{1}{4}-2\right)\cdots\left(\frac{1}{4}-(n-1)\right)(1+x)^{\frac{1}{4}-n}$

The Maclaurin series is

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

where $a_n = \frac{f^n(0)}{n!} = \frac{\frac{1}{4}\left(\frac{1}{4}-1\right)\left(\frac{1}{4}-2\right)\cdots\left(\frac{1}{4}-(n-1)\right)}{n!}$

The Radius of convergence is determined by

$$\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \lim_{n\to\infty}\left|\frac{\frac{\frac{1}{4}\left(\frac{1}{4}-1\right)\left(\frac{1}{4}-2\right)\cdots\left(\frac{1}{4}-(n-1)\right)\left(\frac{1}{4}-n\right)}{(n+1)!}x^{n+1}}{\frac{\frac{1}{4}\left(\frac{1}{4}-1\right)\left(\frac{1}{4}-2\right)\cdots\left(\frac{1}{4}-(n-1)\right)}{(n)!}x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\frac{1}{4}-n}{n+1}\right|\cdot|x| = |x|$$

The Radius of convergence is $|x| < 1$.

(5) The Taylor series for $f(x) = \frac{1}{x}$ is

$$\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots + (-1)^n (x-1)^n + \cdots$$

and has a radius of convergence of $|x-1| < 1$.

Find the Taylor series for $g(x) = \ln(x)$.

Note:

$$\int_1^x \frac{1}{t}\,dt = \ln(t)\Big|_1^x = \ln(x) - \ln(1) = \ln(x)$$

So if I integrate both sides $\int_1^x \cdots dt$ I get $\ln(x)$.

$$\ln(x) = \int_1^x 1\,dt - \int_1^x (t-1)\,dt + \int_1^x (t-1)^2\,dt - \int_1^x (t-1)^3\,dt$$

$$+ \cdots + (-1)^n \int_1^x (t-1)^n\,dt + \cdots$$

Substitute $u = t$ in each integral.

$$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots +$$

$$+ (-1)^n \frac{(x-1)^n}{n} + \cdots$$