## Section C.4

#2 $\frac{dy}{dx} = ky \Rightarrow \frac{1}{y}\frac{dy}{dx} = k$

$$\Rightarrow \ln|y| = kt + C$$

$$\Rightarrow |y| = e^{kt+C}$$

$$\Rightarrow y = \pm e^{C} \cdot e^{kt}$$

Let $A = \pm e^{C} \Rightarrow y = Ae^{kt}$

If $P_1 = (0,4)$ and $P_2 = (1,6)$

$\Rightarrow y(0) = 4 \Rightarrow 4 = Ae^{k(0)} \Rightarrow A = 4$

$\Rightarrow y(1) = 6 \Rightarrow 6 = 4e^{k} \Rightarrow e^{k} = \frac{3}{2}$

$k = \ln(3/2)$

Hence $y(t) = 4e^{\frac{3t}{2}}$

#7 $\frac{dA}{dt} = kA$ where $A(t)$ is the amount of the investment at any time $t$.

As above, $A(t) = Ce^{kt}$ where $C$ and $k$ are constants.
We know that at $t=0$, $A(0) = 2000$

$\Rightarrow 2000 = A(0) = Ce^{k(0)} = C$

We also know that at $t=5$, $A(5) = 2983.65$

So, $2983.65 = 2000\,e^{k5} \Rightarrow e^{k5} = \dfrac{2983.65}{2000}$

$$\Rightarrow 5k = \ln\left[\frac{2983.65}{2000}\right]$$

ie $$k = \frac{1}{5}\ln\left[\frac{2983.65}{2000}\right]$$

<u>Now</u>

$$A(10) = 2000\,e^{\frac{1}{5}\ln\left[\frac{2983.65}{2000}\right]\cdot 10}$$
$$\approx 4{,}451.08$$

#13 $\dfrac{dy}{dx} = ky(L-y)$ Here $0 \le y \le L$.

$$\Rightarrow \frac{1}{y(L-y)}\frac{dy}{dx} = k \Rightarrow \int \frac{1}{y(L-y)}\,dy = kx + C$$

<u>Note:</u> $$\frac{1}{y(L-y)} = \frac{A}{y} + \frac{B}{L-y}$$

$$\Rightarrow 1 = A(L-y) + By$$
$$\Rightarrow 1 = (B-A)y + AL$$

$$\Rightarrow B-A=0 \text{ and } A = 1/L$$

<u>Thus</u>

$$\frac{1}{y(L-y)} = \frac{\frac{1}{L}}{y} + \frac{\frac{1}{L}}{L-y}$$

$$\Rightarrow \frac{1}{L}\ln|y| - \frac{1}{L}\ln|L-y| = kx + C$$

$$\Rightarrow \ln\left|\frac{y}{L-y}\right| = kLx + LC$$

$$\Rightarrow \frac{y}{L-y} = e^{kLx+LC}$$

<u>Note:</u> Both $y \ge 0$ and $L-y \ge 0$.

Now we solve for $y$!

$$y = e^{KLx + LC}(L - y)$$

$$\Rightarrow (1 + e^{KLx+LC})y = L\,e^{KLx+LC}$$

$$\Rightarrow \quad y = \frac{L\,e^{KLx+LC}}{1 + e^{KLx+LC}} = \frac{L}{e^{-KLx-LC} + 1}$$

If we let $A = e^{-LC}$, then

$$y(x) = \frac{L}{1 + A\,e^{-KLx}}.$$

We know $L = 5{,}000$, $P_1 = (0, 250)$ and $P_2 = (25, 2000)$

Using $P_1 \Rightarrow$

$$250 = y(0) = \frac{5{,}000}{1 + A\,e^{-K \cdot 5000 \cdot 0}}$$

$$\Rightarrow \quad 250(1 + A) = 5{,}000$$

$$1 + A = 20$$

$$A = 19$$

Using $P_2 \Rightarrow$

$$2000 = y(25) = \frac{5{,}000}{1 + 19\,e^{-K \cdot 5000 \cdot 25}}$$

$$\Rightarrow \quad 1 + 19\,e^{-K \cdot 5000 \cdot 25} = \frac{5000}{2000} = \frac{5}{2}$$

$$\Rightarrow \quad e^{-K \cdot 5000 \cdot 25} = \frac{3}{2 \cdot 19}$$

$$\Rightarrow \quad -K\cdot 5000\cdot 25 = \ln\left(\frac{3}{2\cdot 19}\right)$$

$$K = \frac{-1}{25\cdot 5000}\ln\left(\frac{3}{2\cdot 19}\right)$$

<u>Hence</u>

$$y(t) = \frac{5{,}000}{1+19\,e^{-5{,}000kt}}$$

with this value of $k$!

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#16 We have that $\frac{dS}{dt} = kS(L-S)$

As in #13 above, we find that

$$S(t) = \frac{L}{1+Ae^{-kLt}}$$

Since $10 = S(0) = \frac{L}{1+A}$, we know that

$$1+A = \frac{L}{10} \qquad A = \frac{L}{10} - 1$$

<u>Hence</u>

$$S(t) = \frac{L}{1+\left(\frac{L}{10}-1\right)e^{-kLt}}$$

#22 We know from example 3, that the solution of

$$\frac{dy}{dt} = k y \ln\left(\frac{L}{y}\right) \quad \text{is}$$

$$y(t) = L e^{-c e^{-kt}} \quad \text{for some constants } c \text{ and } k.$$

If $L = 5000$ and $y(0) = 500$ and $y(1) = 625$, then

$$500 = y(0) \Rightarrow 500 = 5000 e^{-c e^{-k(0)}}$$

$$\Rightarrow \frac{1}{10} = e^{-c}$$

$$\Rightarrow -c = \ln\left(\frac{1}{10}\right) \quad \text{or} \quad c = \ln(10)$$

Moreover if

$$625 = y(1) = 5{,}000 \, e^{-\ln(10) e^{-k}}$$

$$\Rightarrow e^{-\ln(10) e^{-k}} = \frac{625}{5{,}000}$$

$$\Rightarrow -\ln(10) e^{-k} = \ln\left[\frac{625}{5{,}000}\right]$$

$$e^{-k} = \frac{-1}{\ln(10)} \ln\left(\frac{625}{5{,}000}\right)$$

$$-k = \ln\left[\frac{-1}{\ln(10)} \ln\left(\frac{625}{5{,}000}\right)\right]$$

Plugging this in we find that

$$y(t) = 5{,}000\, e^{-\ln(10)\, e^{\ln\left[\frac{-1}{\ln(10)} \ln\left[\frac{625}{5{,}000}\right]\right] t}}$$

#23 Let $y(t)$ be the population of beavers after $t$ years

By Example 3, we know that

$$y(t) = L\, e^{-c e^{-kt}}$$

Here $L = 60$. We know that $y(0) = 8$ and $y(3) = 15$

$$\Rightarrow \quad 8 = y(0) = 60\, e^{-c e^{-k(0)}}$$

$$\Rightarrow \quad e^{-c} = \frac{8}{60} = \frac{4}{30} = \frac{2}{15}$$

$$\Rightarrow \quad -c = \ln(2/15)$$

<u>and</u>

$$15 = y(3) = 60\, e^{\ln(2/15)\, e^{-3k}}$$

$$\Rightarrow \quad e^{\ln(2/15) \cdot e^{-3k}} = \frac{15}{60} = \frac{1}{4}$$

$$\Rightarrow \quad \ln\left(\frac{2}{15}\right) e^{-3k} = \ln\left(\frac{1}{4}\right)$$

$$\Rightarrow \quad -3k = \ln\left[\frac{\ln\left(\frac{1}{4}\right)}{\ln\left(\frac{2}{15}\right)}\right]$$

Then

$$y(t) = 60\, e^{\ln(\frac{2}{15})\, e^{\frac{t}{3}\ln\left(\frac{\ln(\frac{1}{4})}{\ln(2/15)}\right)}}$$

$$\Rightarrow \quad y(10) = 60\, e^{\ln(2/15)\, e^{10/3\,\ln\left(\frac{\ln(1/4)}{\ln(2/15)}\right)}}$$

$$\approx 34$$

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#33 $\dfrac{dP}{dt} - kP = N$ 1st order linear

The integrating factor is

$$u(t) = e^{\int -k\,dt} = e^{-kt}$$

$$\Rightarrow \quad P(t) = \frac{1}{u(t)}\left[\int N e^{-kt}\,dt + C\right]$$

$$= e^{kt}\left[-\frac{N}{k}e^{-kt} + C\right]$$

$$= -\frac{N}{k} + C e^{kt}$$

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#35 Just as above

$$A(t) = -\frac{P}{r} + C e^{rt}$$

If $0 = A(0) = -\frac{P}{r} + C e^{r(0)}$

$$\Rightarrow \quad C = \frac{P}{r} \qquad \text{Hence} \quad A(t) = \frac{P}{r}\left(-1 + e^{rt}\right).$$