Key to Test 1

1 Differentiate both sides of

$$y^2 = 2cx.$$

We get

$$2yy' = 2c$$

Multiply both sides by x

$$\Rightarrow \quad x2yy' = 2cx = y^2$$

so

$$y(2xy' - y) = 0.$$

Either $y = 0$ or $2xy' - y = 0$ ✓

2. Find the general solution of

$$x^3y' + e^{-x^2} = -4x^2y \quad \text{for } x > 0.$$

Put in standard form:

$$x^3y' + 4x^2y = -e^{-x^2}$$

$$\Rightarrow \quad y' + \frac{4}{x}y = \frac{-1}{x^3}e^{-x^2}$$

Here

$$P(x) = 4/x \qquad \text{and} \qquad Q(x) = -\frac{1}{x^3} e^{-x^2}$$

The integrating factor is

$$u(x) = e^{\int P(x)dx} = e^{4\int \frac{1}{x} dx}$$

$$= e^{4\ln|x|}$$

$$= e^{\ln(x^4)} = x^4$$

So $u(x) = x^4$.

The solution is then

$$y(x) = \frac{1}{u(x)} \left[\int Q(x) u(x)\, dx + C \right]$$

$$= \frac{1}{x^4} \left[-\int \frac{1}{x^3} e^{-x^2} \cdot x^4\, dx + C \right]$$

$$= \frac{1}{x^4} \left[-\int x e^{-x^2}\, dx + C \right]$$

Let $u = -x^2$

$du = -2x\,dx$

$u = -x^2 \quad du = -2x\,dx$

$$-\int x e^{-x^2} dx = +\frac{1}{2}\int e^{u} du = +\frac{1}{2} e^{-x^2} + C$$

$\Rightarrow$

$$y(x) = +\frac{1}{2x^4} e^{-x^2} + \frac{C}{x^4}.$$

3 $$\frac{dy}{dx} = \frac{\sin(2x)}{\cos(3y)}$$

a) general solution

$$\cos(3y)\frac{dy}{dx} = \sin(2x)$$

$$\Rightarrow \int \cos(3y)\,dy = \int \sin(2x)\,dx$$

$$\frac{1}{3}\sin(3y) = -\frac{1}{2}\cos(2x) + C$$

b) Particular solution satisfying $y(\pi/2) = \pi/3$

$$\Rightarrow \frac{1}{3}\sin(3(\pi/3)) = -\frac{1}{2}\cos(2(\pi/2)) + C$$

$$0 = -\frac{1}{2}(-1) + C$$

$$C = -1/2$$

$$\Rightarrow \frac{1}{3}\sin(3y) = -\frac{1}{2}\cos(2x) - 1/2$$

4 a) $$\frac{dP}{dt} = K(L-P)(L+P)$$

$$\Rightarrow \quad \frac{1}{(L-P)(L+P)} \frac{dP}{dt} = K$$

$$\Rightarrow \quad \int \frac{1}{(L-P)(L+P)} \, dP = Kt + C$$

Note:

$$\frac{1}{(L-P)(L+P)} = \frac{A}{L-P} + \frac{B}{L+P}$$

$$\Rightarrow \quad 1 = A(L+P) + B(L-P)$$

$$\Rightarrow \quad 1 = (A-B)P + AL + BL$$

$$\Rightarrow \quad A - B = 0 \quad \text{and} \quad 1 = AL + BL$$

Hence $A = B = \frac{1}{2L}$.

Thus

$$\int \frac{1}{(L-P)(L+P)} \, dP = \frac{1}{2L} \int \frac{1}{L-P} dP + \frac{1}{2L} \int \frac{1}{L+P} dP$$

$$= -\frac{1}{2L} \ln|L-P| + \frac{1}{2L} \ln|L+P|$$

So

$$\frac{1}{2L} \ln\left|\frac{L+P}{L-P}\right| = Kt + C$$ is the general solution.

b) If $L=10$ and $P(0)=0$, then

$$\frac{1}{2\cdot 10} \ln\left|\frac{10+0}{10-0}\right| = K(0) + C$$

$$\Rightarrow \quad \frac{1}{20}\ln(1) = C$$

$$\Rightarrow \quad C = 0.$$

If $P(2) = 5$, then

$$\frac{1}{20}\ln\left|\frac{10+5}{10-5}\right| = 2K$$

$$\Rightarrow K = \frac{1}{40}\ln(3)$$

$$\Rightarrow \boxed{\frac{1}{20}\ln\left|\frac{10+P}{10-P}\right| = \frac{1}{40}\ln(3)\,t}$$

(5)

$$(x+y+z)^2 = 2(xy+xz+yz-1)+27$$

$$\Rightarrow x^2+xy+xz+yx+y^2+yz+zx+zy+z^2 = 2xy+2xz+2yz-2+27$$

Much cancels!

$$\Rightarrow x^2+y^2+z^2 = 25$$

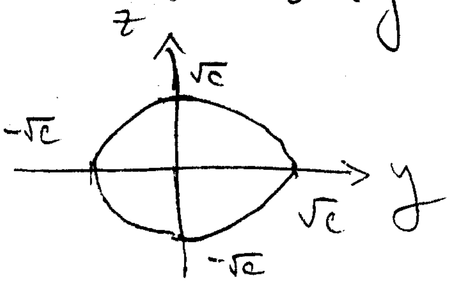
Center $C=(0,0,0)$ radius $r=5$.

6 a) Consider $x = z^2+y^2$

If $x=c$, then we have the equation

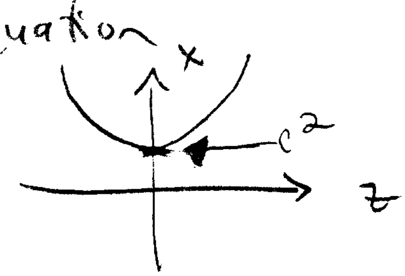
$$c = z^2+y^2$$

which is a circle of radius $\sqrt{c}$



If $y=c$, then we have the equation

$x = z^2+c^2$ which is a parabola



If $z=c$, then we have the equation

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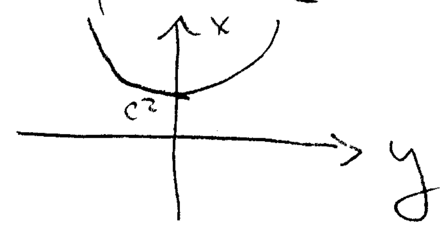
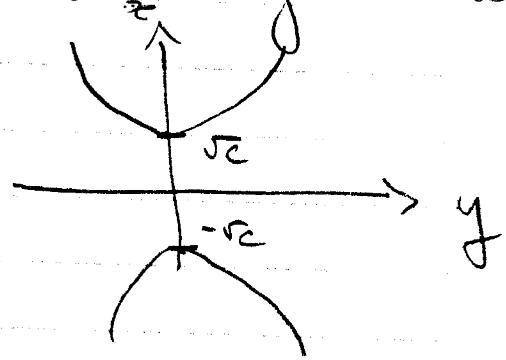


b) Consider $x = z^2 - y^2$

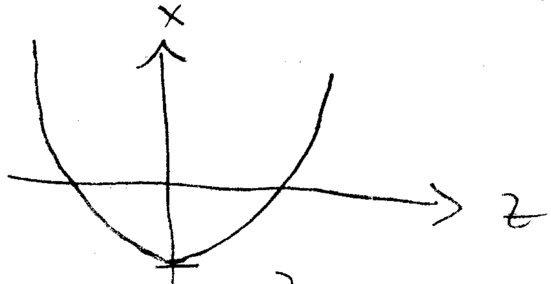
If $x=c$, then we have the equation

$c = z^2 - y^2$ which is a hyperbola.



If $y=c$, then we have the equation

$x = z^2 - c^2$ which is a parabola



If $z=c$, then we have the equation

$x = -y^2 + c^2$ which is a parabola