(1) Consider the function

$$f(x,y) = \ln\left[64 - 4(x-1)^2 - 16(y+2)^2\right].$$

a) Sketch the domain of f.

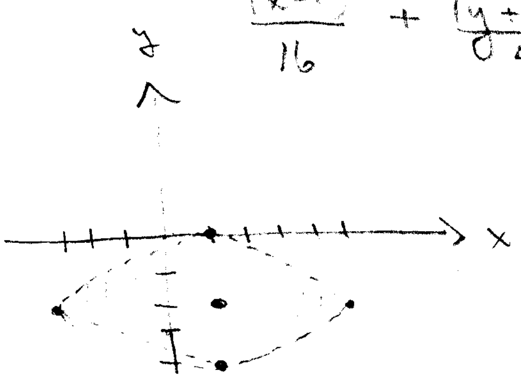
b) Sketch the level curves corresponding to $f(x,y) = 0$ and $f(x,y) = \ln[48]$ on the same graph.

a) The domain is the set of points $P = (x,y)$ for which

$$64 - 4(x-1)^2 - 16(y+2)^2 > 0$$

or $$4(x-1)^2 + 16(y+2)^2 < 64$$

i.e. $$\frac{(x-1)^2}{16} + \frac{(y+2)^2}{4} < 1$$



outer ellipse not included!

$\Rightarrow$ $0 = \ln\left[64 - 4(x-1)^2 - 16(y+2)^2\right]$

$\Rightarrow$ $1 = 64 - 4(x-1)^2 - 16(y+2)^2$

$4(x-1)^2 + 16(y+2)^2 = 63$

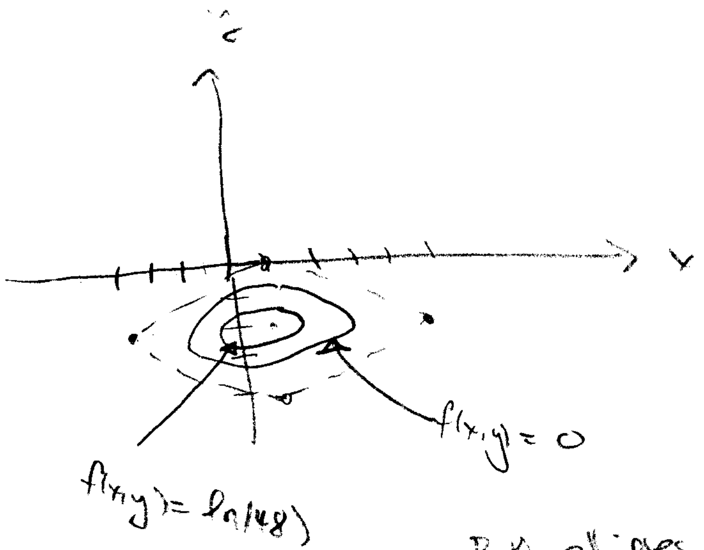
$$\frac{(x-1)^2}{\frac{63}{4}} + \frac{(y+2)^2}{\frac{63}{16}} = 1$$

$\ln(48) = \ln\left[64 - 4(x-1)^2 - 16(y+2)^2\right]$

$\Rightarrow$ $48 = 64 - 4(x-1)^2 - 16(y+2)^2$

$4(x-1)^2 + 16(y+2)^2 = 16$

$$\frac{(x-1)^2}{4} + (y+2)^2 = 1$$



Both ellipses about P = (1,-2).

(2) Consider the following equation

$$-15z^2 - 25x + 10y^2 = -5.$$

Sketch the traces corresponding to:

a) $x = 1$

b) $y = 1$

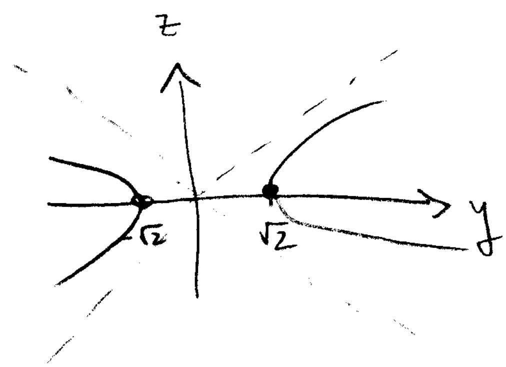
c) $z = 1$
Be sure to label the axes.

a) $\boxed{x=1}$

$$-15z^2 - 25 + 10y^2 = -5$$

$$-15z^2 + 10y^2 = 20$$

$$\frac{y^2}{2} - \frac{z^2}{\frac{4}{3}} = 1$$



hyperbola

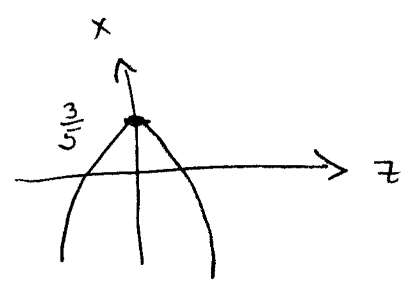
b) $\boxed{y=1}$

$$-15z^2 - 25x + 10 = -5$$

$$25x = -15z^2 + 15$$

$$x = -\frac{3}{5}z^2 + \frac{3}{5}$$

parabola

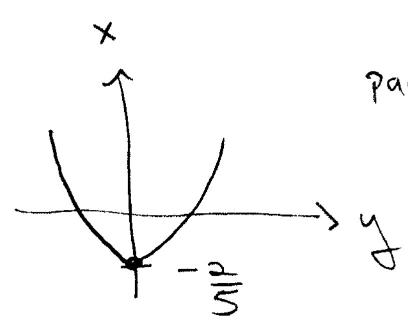


c) $\boxed{z=1}$

$$-15 - 25x + 10y^2 = -5$$

$$25x = 10y^2 - 10$$

$$x = \frac{2}{5}y^2 - \frac{2}{5}$$



parabola

(3) Find all first order partial derivatives of

$$f(x,y,z) = z\tan\left(2\sqrt[3]{xz}\,e^{xy}\right).$$

$$\frac{\partial f}{\partial x}(x,y,z) = z\sec^2\left(2\sqrt[3]{xz}\,e^{xy}\right)\cdot\frac{\partial}{\partial x}\left(2\sqrt[3]{xz}\,e^{xy}\right)$$

$$= z\sec^2\left(2\sqrt[3]{xz}\,e^{xy}\right)\left[2\cdot\frac{1}{3}(xz)^{-2/3}\cdot z\,e^{xy} + 2\sqrt[3]{xz}\,e^{xy}\cdot y\right]$$

$$\frac{\partial f}{\partial y}(x,y,z) = z\sec^2\left(2\sqrt[3]{xz}\,e^{xy}\right)\frac{\partial}{\partial y}\left(2\sqrt[3]{xz}\,e^{xy}\right)$$

$$= z\sec^2\left(2\sqrt[3]{xz}\,e^{xy}\right)2\sqrt[3]{xz}\,e^{xy}\cdot x$$

$$\frac{\partial f}{\partial z}(x,y,z) = \tan\left(2\sqrt[3]{xz}\,e^{xy}\right) + z\sec^2\left(2\sqrt[3]{xz}\,e^{xy}\right)\frac{\partial}{\partial z}\left(2\sqrt[3]{xz}\,e^{xy}\right)$$

$$= \tan\left(2\sqrt[3]{xz}\,e^{xy}\right) + z\sec^2\left(2\sqrt[3]{xz}\,e^{xy}\right)2e^{xy}\cdot\frac{1}{3}(xz)^{-2/3}\cdot x$$

(4) Find and classify all critical points of

$$f(x,y) = 3xy - x^2y - xy^2.$$

$f_x(x,y) = 3y - 2xy - y^2 = y(3-2x-y)$
$f_y(x,y) = 3x - x^2 - 2xy = x(3-x-2y)$

Both always exist.

Only critical points occur when $f_x = f_y = 0$.

$f_x = 0 \Rightarrow$ either $y = 0$ or $3-2x-y = 0$

$f_y = 0 \Rightarrow$ either $x = 0$ or $3-x-2y = 0$.

Case i) Suppose $y = 0$.

Then either $x = 0$ or $3 - x - 2(0) = 0$ i.e. $x = 3$

$P_1 = (0,0)$, $P_2 = (3,0)$

Case ii) Suppose $3-2x-y = 0$

Then either $x = 0$, in which case $3 - 2(0) - y = 0$ i.e. $y = 3$

$P_3 = (0,3)$

or $3 - x - 2y = 0$ i.e. $x = 3 - 2y \Rightarrow 3 - 2(3-2y) - y = 0$

$3 - 6 + 4y - y = 0$

$3y = 3$

$y = 1$

$x = 3 - 2 = 1$

$P_4 = (1,1)$

Now

$f_{xx}(x,y) = -2y$

$f_{yy}(x,y) = -2x$

$f_{xy}(x,y) = 3-2x-2y$

So that

$$
\begin{aligned}
d(x,y) &= f_{xx}(x,y)f_{yy}(x,y) - (f_{xy}(x,y))^2 \\
&= (-2y)(-2x) - (3-2x-2y)^2 \\
&= 4xy - (3-2x-2y)^2
\end{aligned}
$$

For $P_1 = (0,0)$

$d(0,0) = -9 < 0 \Rightarrow$ saddle point

For $P_2 = (3,0)$

$d(3,0) = 0-(3-6)^2 = -9 < 0 \Rightarrow$ saddle point

For $P_3 = (0,3)$

$d(0,3) = 0-(3-6)^2 = -9 < 0 \Rightarrow$ saddle point

For $P_4 = (1,1)$

$d(1,1) = 4-(3-2-2)^2 = 3 > 0$

$f_{xx}(1,1) = -2 < 0 \Rightarrow$ relative max.

(5) Find the maximum value of the function

$$f(x,y) = \frac{1}{x} + \frac{1}{y}$$

subject to the constraint

$$\frac{1}{x^2} + \frac{1}{y^2} = 1.$$

Assume both $x > 0$ and $y > 0$.

Set

$$F(x,y,\lambda) = \frac{1}{x} + \frac{1}{y} - \lambda\left(\frac{1}{x^2} + \frac{1}{y^2} - 1\right)$$

$$F_x(x,y,\lambda) = -\frac{1}{x^2} + 2\lambda\frac{1}{x^3} \qquad F_x = 0 \Rightarrow \frac{2\lambda}{x^3} = \frac{1}{x^2} \Rightarrow x = 2\lambda$$

$$F_y(x,y,\lambda) = -\frac{1}{y^2} + 2\lambda\frac{1}{y^3} \qquad F_y = 0 \Rightarrow \frac{2\lambda}{y^3} = \frac{1}{y^2} \Rightarrow y = 2\lambda$$

$$F_\lambda(x,y,\lambda) = -\left(\frac{1}{x^2} + \frac{1}{y^2} - 1\right) \qquad F_\lambda = 0 \Rightarrow \frac{1}{x^2} + \frac{1}{y^2} = 1$$

$$\Rightarrow \frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} = 1$$

$$\Rightarrow 4\lambda^2 = 2$$

$$\lambda^2 = \frac{1}{2}$$

$$\lambda = \pm\sqrt{\frac{1}{2}}$$

as $x, y > 0$

Take $\lambda > 0$

$$x = \frac{2}{\sqrt{2}} \qquad y = \frac{2}{\sqrt{2}}$$

$$f\left(\frac{2}{\sqrt{2}}, \frac{2}{\sqrt{2}}\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$$

(6) Calculate the following integral

$$\int_0^1 \int_0^{x^2} \left(xe^y + \sin\left(\frac{\pi y}{x^2}\right)\right) dy\, dx\,.$$

The inner integral is

$$\int_0^{x^2} xe^y + \sin\left(\frac{\pi y}{x^2}\right) dy = xe^y \Big|_0^{x^2} - \frac{\cos\left(\frac{\pi y}{x^2}\right)}{\frac{\pi}{x^2}} \Bigg|_0^{x^2}$$

$$= xe^{x^2} - x - \left(\frac{x^2}{\pi}\cos(\pi) - \frac{x^2}{\pi}\cos(0)\right)$$

$$= xe^{x^2} - x + \frac{2}{\pi}x^2$$

Thus

$$\int_0^1 \int_0^{x^2} \left(xe^y + \sin\left(\frac{\pi y}{x^2}\right)\right) dy\, dx$$

$$= \int_0^1 xe^{x^2} - x + \frac{2}{\pi}x^2\, dx$$

$$= \int_0^1 xe^{x^2} dx - \frac{x^2}{2}\Big|_0^1 + \frac{2}{\pi}\frac{x^3}{3}\Big|_0^1$$

$u = x^2$

$du = 2x\,dx$

$$= \frac{1}{2}\int_0^1 e^u du - \frac{1}{2} + \frac{2}{3\pi}$$

$$= \frac{1}{2}(e-1) - \frac{1}{2} + \frac{2}{3\pi}$$