Key To Test 3

(1)

Consider

$$a_n = \frac{\cos(3n^2+2n-1)}{4n-2}$$

a)

Note: The numerator is a cosine and therefore bounded (in absolute value) by 1. As the denominator goes to infinity as $n \to \infty$, the limit of the sequence is zero.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\cos(3n^2+2n-1)}{4n-2} = 0.$$

b) Since the limit of the terms is zero, I cannot use the n-th term test. Thus, I cannot say anything about the convergence or divergence of this series.

2 Consider

$$\sum_{n=3}^{\infty} 2\left(-\frac{3}{7}\right)^n$$

a) This series is geometric with $a=2$ and $r = -3/7$.

As $|r| = \frac{3}{7} < 1$, the series converges.

b) If we sum the "entire geometric series" we get

$$\sum_{n=0}^{\infty} 2\left(\frac{-3}{7}\right)^n = \frac{2}{1-\left(\frac{-3}{7}\right)} = \frac{2}{\frac{10}{7}} = \frac{7}{5}$$

To calculate the above though, we get

$$\sum_{n=3}^{\infty} 2\left(-\frac{3}{7}\right)^n = \sum_{n=0}^{\infty} 2\left(\frac{-3}{7}\right)^n - \left[2 + 2\left(\frac{-3}{7}\right) + 2\left(\frac{-3}{7}\right)^2\right]$$

$$= \frac{7}{5} - 2 + \frac{6}{7} - \frac{18}{49}$$

c) $$\sum_{n=0}^{N} 2\left(-\frac{3}{7}\right)^n = \frac{2\left(1-\left(\frac{-3}{7}\right)^{N+1}\right)}{1-\left(\frac{-3}{7}\right)} = \frac{7}{5}\left(1-\left(\frac{-3}{7}\right)^{N+1}\right)$$

Consider

a) $$\sum_{n=1}^{\infty} \frac{1}{n^{\pi^2/9}}$$

This is a p-series with $p = \pi^2/9 > 1$. This series converges.

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b) $$\sum_{n=1}^{\infty} \frac{(n+1)9^n}{n\,7^{2n}}$$ Here the terms are $a_n = \frac{(n+1)9^n}{n\,7^{2n}}$.

Use Ratio Test!

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+2)\,9^{n+1}}{(n+1)\,7^{2(n+1)}}}{\frac{(n+1)\,9^n}{n\,7^{2n}}} = \frac{n+2}{(n+1)} \cdot \frac{n}{(n+1)} \frac{9^{n+1}}{9^n} \cdot \frac{7^{2n}}{7^{2n+2}}$$

$$\Rightarrow \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{9}{7^2} = \frac{9}{49} < 1$$

So the series converges.

c) $$\sum_{n=1}^{\infty} \frac{(2n+5)!}{(3n)!\,10^n}$$ Here the terms are $a_n = \frac{(2n+5)!}{(3n)!\,10^n}$

Ratio Test

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(2(n+1)+5)!}{(3(n+1))!\,10^{n+1}}}{\frac{(2n+5)!}{(3n)!\,10^n}} = \frac{(2n+7)!}{(2n+5)!} \frac{(3n)!\;10^n}{(3n+3)!\;10^{n+1}}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(2n+7)(2n+6)}{(3n+3)(3n+2)(3n+1)\,10} \right| = 0 < 1$$

$\boxed{\text{So the series converges}}$

Consider

$$f(x) = \sqrt[5]{x+2} = (x+2)^{1/5}$$

a) Find the Taylor series centered at -1.

The series is

$$\sum_{n=0}^{\infty} a_n (x+1)^n \quad \text{where} \quad a_n = \frac{f^{(n)}(-1)}{n!}$$

Calculate the derivatives.

$$f(x) = (x+2)^{1/5}$$

$$f'(x) = \tfrac{1}{5}(x+2)^{1/5-1}$$

$$f''(x) = \tfrac{1}{5}\left(\tfrac{1}{5}-1\right)(x+2)^{\frac{1}{5}-2}$$

$$f'''(x) = \tfrac{1}{5}\left(\tfrac{1}{5}-1\right)\left(\tfrac{1}{5}-2\right)(x+2)^{\frac{1}{5}-3}$$

$$\vdots$$

$$f^{(n)}(x) = \tfrac{1}{5}\left(\tfrac{1}{5}-1\right)\cdots\left(\tfrac{1}{5}-(n-1)\right)(x+2)^{\frac{1}{5}-n}$$

The coefficients are then

$$a_n = \frac{f^{(n)}(-1)}{n!}$$

$$= \frac{\tfrac{1}{5}\left(\tfrac{1}{5}-1\right)\cdots\left(\tfrac{1}{5}-(n-1)\right)(-1+2)^{\frac{1}{5}-n}}{n!}$$

$$= \frac{\tfrac{1}{5}\left(\tfrac{1}{5}-1\right)\cdots\left(\tfrac{1}{5}-(n-1)\right)}{n!}$$

Plugging in gives the series.

For the radius of convergence we use the ratio test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}(x+1)^{n+1}}{a_n(x+1)^n}\right| = \lim_{n\to\infty}\left|\frac{\frac{\frac{1}{5}\left(\frac{1}{5}-1\right)\cdots\left(\frac{1}{5}-(n-1)\right)\left(\frac{1}{5}-n\right)}{(n+1)!}(x+1)^{n+1}}{\frac{\frac{1}{5}\left(\frac{1}{5}-1\right)\cdots\left(\frac{1}{5}-(n-1)\right)}{n!}(x+1)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\left(\frac{1}{5}-n\right)(x+1)}{(n+1)}\right| = |x+1|$$

So the radius of convergence is $|x+1| < 1$

(5) Find the Taylor polynomial (centered at 0) with degree 13 for

$$f(x) = \frac{3\sin(2x^2)}{5x}$$

Use the result for $\sin(x)$ but evaluate at $2x^2$

$$f(x) = \frac{3}{5x} \sum_{n=0}^{\infty} (-1)^n \frac{(2x^2)^{2n+1}}{(2n+1)!}$$

write out some terms

$$= \frac{3}{5x}\left[2x^2 + (-1)\frac{(2x^2)^3}{3!} + \frac{(2x^2)^5}{5!} + (-1)\frac{(2x^2)^7}{7!} + \cdots \right]$$

$$= \frac{3}{5x}\left[2x^2 - \frac{2^3 x^6}{3!} + \frac{2^5 x^{10}}{5!} - \frac{2^7 x^{14}}{7!} + \cdots \right]$$

Divide

$$= \underbrace{\frac{6}{5}x - \frac{3\cdot 2^3}{5\cdot 3!}x^5 + \frac{3\cdot 2^5}{5\cdot 5!}x^9 - \frac{3\cdot 2^7}{5\cdot 7!}x^{13}} + \cdots$$

This is the Taylor Polynomial of degree 13.