Homework Set No. 1 – Probability Theory (235A), Fall 2009

Due: 10/6/09

1. (a) If $(\Omega, \mathcal{F}, \mathbf{P})$ is a probability space and $A, B \in \mathcal{F}$ are events such that $\mathbf{P}(B) \neq 0$, the **conditional probability of** A **given** B is denoted $\mathbf{P}(A|B)$ and defined by

$$\mathbf{P}(A|B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)}.$$

Prove the **total probability formula:** if $A, B_1, B_2, \ldots, B_k \in \mathcal{F}$ such that Ω is the disjoint union of $B_1, \ldots, B_k$ and $\mathbf{P}(B_i) \neq 0$ for $1 \leq i \leq k$, then

$$\mathbf{P}(A) = \sum_{i=1}^{k} \mathbf{P}(B_i)\mathbf{P}(A|B_i). \tag{TPF}$$

(b) An urn initially contains one white ball and one black ball. At each step of the experiment, a ball is drawn at random from the urn, then put back and another ball of the same color is added. Prove that the number of white balls that are in the urn after N steps is a uniform random number in $\{1, 2, \ldots, N+1\}$. That is, the event that the number of white balls after step N is equal to k has probability $1/(N+1)$ for each $1 \leq k \leq N+1$. (Note: The idea is to use (TPF), but there is no need to be too formal about constructing the relevant probability space - you can assume an intuitive notion of probabilities.)

2. If $\Omega = \{1, 2, 3\}$, list all the possible σ-algebras of subsets of Ω.

3. Let $(\Omega, \mathcal{F})$ be a measurable space. A **pre-probability measure** is a function $\mathbf{P} : \mathcal{F} \to [0, 1]$ that satisfies

$$\mathbf{P}(\emptyset) = 0, \qquad \mathbf{P}(\Omega) = 1. \tag{P1}$$

If a pre-probability measure $\mathbf{P}$ satisfies

$$A_1, A_2, \ldots \in \mathcal{F} \text{ are pairwise disjoint} \implies \mathbf{P}(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mathbf{P}(A_n). \tag{P2}$$

then we say that it is σ**-additive**. If it satisfies

$$A_1, \dots, A_n \in \mathcal{F} \text{ are pairwise disjoint } \implies \mathbf{P}(\cup_{k=1}^n A_k) = \sum_{k=1}^n \mathbf{P}(A_k). \tag{P3}$$

then we say that it is **additive**. Recall that we defined a **probability measure** to be a pre-probability measure that is σ-additive.

We say that a pre-probability measure satisfies the **continuity** properties if it satisfies

$$(A_n)_{n=1}^\infty \subset \mathcal{F}, \quad A_n \subset A_{n+1}\ \forall n \implies \mathbf{P}(\cup_{n=1}^\infty A_n) = \lim_{n\to\infty} \mathbf{P}(A_n), \tag{CONT1}$$

$$(A_n)_{n=1}^\infty \subset \mathcal{F}, \quad A_n \supset A_{n+1}\ \forall n \implies \mathbf{P}(\cap_{n=1}^\infty A_n) = \lim_{n\to\infty} \mathbf{P}(A_n). \tag{CONT2}$$

Prove that a probability measure satisfies the continuity properties, and that an additive pre-probability measure that satisfies the first continuity property (CONT1) ("continuity from below") is σ-additive, and is therefore a probability measure.

4. (a) A coin has some bias $p \in (0,1)$, so when tossed it comes up Heads with probability p, or Tails with probability $1-p$. Suppose the coin is tossed N times independently, and let $A_{N,k}$ denote the event that the result came up Heads exactly k times. Refresh your memory concerning why the Binomial Distribution Formula, which says that

$$\mathbf{P}(A_{N,k}) = \binom{N}{k} p^k (1-p)^{N-k}, \tag{BINOM}$$

is true. You may submit a short written explanation to test your understanding, but it is not required.

(b) A group of N prisoners is locked up in a prison, each in a separate cell with no ability to communicate with the other prisoners. Each cell contains a mysterious on/off electrical switch. One evening the warden visits each of the prisoners and presents them with the following dilemma: During the night each prisoner must choose whether to leave his switch in the on or off position. If at midnight *exactly one* of the switches is in the on position, all the prisoners will be set free in the morning; otherwise they will all be executed!

The prisoners cannot coordinate their actions, but they are all rational, know calculus and probability theory, and each is equipped with a random number generator. Find the strategy that the prisoners will take to maximize their chance of survival, and compute what that chance is, as a function of N and in the limit when N is very large. For extra fun, try to guess in advance how big or small you expect the survival likelihood to be, and see how your guess measures up to the actual result.

5. (i) Let Ω be a set, and let $\mathcal{S} = \{\mathcal{F}_i\}_{i\in I}$ be some collection of σ-algebras of subsets of Ω, indexed by some index set I (note that $\mathcal{S}$ is a *set* of *subsets* of *subsets* of Ω - try to avoid dizziness!). Prove that the intersection of all the $\mathcal{F}_i$'s (i.e., the collection of subsets of Ω that are elements of all the $\mathcal{F}_i$'s) is also a σ-algebra.

(ii) Let Ω be a set, and let $\mathcal{A}$ be a collection of subsets of Ω. Prove that there exists a unique σ-algebra $\sigma(\mathcal{A})$ of subsets of Ω that satisfies the following two properties:

1. $\mathcal{A} \subset \sigma(\mathcal{A})$ (in words, $\sigma(\mathcal{A})$ contains all the elements of $\mathcal{A}$).
2. $\sigma(\mathcal{A})$ is the minimal σ-algebra satisfying property 1 above, in the sense that if $\mathcal{F}$ is any other σ-algebra that contains all the elements of $\mathcal{A}$, then $\sigma(\mathcal{A}) \subset \mathcal{F}$.

The σ-algebra $\sigma(\mathcal{A})$ is called the **σ-algebra generated by $\mathcal{A}$**.

Hint for (ii): Let $(\mathcal{F}_i)_{i\in I}$ be the collection of all σ-algebras of subsets of Ω that contain $\mathcal{A}$. This is a non-empty collection, since it contains for example $\mathcal{P}(\Omega)$, the set of all subsets of Ω. Any σ-algebra $\sigma(\mathcal{A})$ that satisfies the two properties above is necessarily a subset of any of the $\mathcal{F}_i$'s, hence it is also contained in the intersection of all the $\mathcal{F}_i$'s, which is a σ-algebra by part (i) of the question.