Homework Set No. 6 – Probability Theory (235A), Fall 2009

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1. Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Prove that

$$\int_0^1 \int_0^1 \ldots \int_0^1 f\left(\frac{x_1 + x_2 + \ldots + x_n}{n}\right) dx_1\, dx_2 \ldots dx_n \xrightarrow[n\to\infty]{} f(1/2).$$

2. A bowl contains n spaghetti noodles arranged in a chaotic fashion. Bob performs the following experiment: he picks two random ends of noodles from the bowl (chosen uniformly from the $2n$ possible ends), ties them together, and places them back in the bowl. Then he picks at random two more ends (from the remaining $2n-2$), ties them together and puts them back, and so on until no more loose ends are left.

Let L_n denote the number of **spaghetti loops** at the end of this process (a loop is a chain of one or more spaghettis whose ends are tied to each other to form a cycle). Compute $\mathbf{E}(L_n)$ and $\mathbf{V}(L_n)$. Find a sequence of numbers $(b_n)_{n=1}^{\infty}$ such that

$$\frac{L_n}{b_n} \xrightarrow[n\to\infty]{\mathbf{P}} 1,$$

if such a sequence exists.

3. Martians communicate in a binary language with two symbols, 0 and 1. A text of length n symbols written in the Martian language looks like a sequence $X_1, X_2, \ldots, X_n$ of i.i.d. random symbols, each of which is 1 with probability p and 0 with probability $1-p$. Here, $p \in (0,1)$ is a parameter (the "Martian bias").

Define the **entropy function** $H(p)$ by

$$H(p) = -p\log_2 p - (1-p)\log_2(1-p).$$

Prove the following result that effectively says that if n is large, then with high probability a Martian text of length n can be encoded into an ordinary (man-made) computer file of length approximately $n \cdot H(p)$ computer bits (note that if $p \neq 1/2$ then this is smaller than n, meaning that the text can be compressed by a linear factor):

Theorem. *Let $X_1, X_2, X_3, \ldots$ be a sequence of i.i.d. Martian symbols (i.e., Bernoulli variables with bias p). Denote by $\boldsymbol{T}_n = (X_1, \ldots, X_n)$ the Martian text comprised of the* *first n symbols. For any $\epsilon > 0$, if n is sufficiently large, the set $\{0,1\}^n$ of possible texts of length n can be partitioned into two disjoint sets,*

$$\{0,1\}^n = A_n \cup B_n,$$

such that the following statements hold:

1. $\mathbf{P}(\boldsymbol{T}_n \in B_n) < \epsilon$
2. $2^{n(H(p)-\epsilon)} \leq |A_n| \leq 2^{n(H(p)+\epsilon)}$.

Notes: The texts in B_n can be thought of as the "exceptional sequences" – they are the Martian texts of length n that are rarely observed. The texts in A_n are called "typical sequences". Because of the two-sided bounds the theorem gives on the number of typical sequences, it follows that we can encode them in a computer file of size approximately $nH(p)$ bits, provided we prepare in advance a "code" that translates the typical sequences to computer files of the appropriate size (this can be done algorithmically, for example by making a list of typical sequences sorted in lexicographic order, and matching them to successive binary strings of length $(H(p)+\epsilon)n$).

Hint: To prove the theorem, let P_n be the random variable given by

$$P_n = \prod_{k=1}^{n} \left(p^{X_k}(1-p)^{1-X_k} \right).$$

Note that P_n measures the probability of the sequence that was observed up to time n. (Somewhat unusually, in this problem the probability itself is thought of as a random variable). Try to represent P_n in terms of cumulative sums of a sequence of i.i.d. random variables. Apply the Weak Law of Large Numbers to that sequence, and see where that gets you.

4. Prove the following one-sided version of Chebyshev's inequality: For any r.v. X and $t \geq 0$,

$$\mathbf{P}(X - \mathbf{E}X \geq t) \leq \frac{\sigma^2(X)}{t^2 + \sigma^2(X)}.$$

Hint: Assume without loss of generality that $\mathbf{E}X = 0$. For any $a > 0$, we have that $\mathbf{P}(X \geq t) \leq \mathbf{P}((X+a)^2 \geq (a+t)^2)$. Bound this using known methods and then look for the value of a that gives the best bound.

5. Let $X_1, X_2, \dots$ be a sequence of i.i.d. r.v.'s with distribution $\mathrm{Exp}(1)$. Prove that

$$\mathbf{P}\left(\limsup_{n\to\infty} \frac{X_n}{\log n} = 1\right) = 1.$$

6. (Optional). Let $A = (X_{i,j})_{i,j=1}^n$ be a random $n \times n$ matrix of i.i.d. random signs (i.e., random variables such that $\mathbf{P}(X_{i,j} = -1) = \mathbf{P}(X_{i,j} = 1) = 1/2$). Compute $\mathrm{Var}(\det(A))$.