Homework Set No. 7 – Probability Theory (235A), Fall 2009

Posted: 11/10/09 — Due: 11/17/09

1. **(a)** Read, in Durrett's book (p. 63 in the 3rd edition) or on Wikipedia, the statement and proof of **Kronecker's lemma**.

(b) Deduce from this lemma, using results we learned in class, the following rate of convergence result for the Strong Law of Large Numbers in the case of a finite variance: If $X_1, X_2, \ldots$ is an i.i.d. sequence such that $\mathbf{E}X_1 = 0$, $\mathbf{V}(X_1) < \infty$, and $S_n = \sum_{k=1}^n X_k$, then for any $\epsilon > 0$,

$$\frac{S_n}{n^{1/2+\epsilon}} \xrightarrow[n\to\infty]{\text{a.s.}} 0.$$

Notes. When X_1 is a "random sign", i.e., a random variable that takes the values $-1, +1$ with respective probabilities $1/2, 1/2$, the sequence of cumulative sums $(S_n)_{n=1}^\infty$ is often called a **(symmetric) random walk on** $\mathbb{Z}$, since it represents the trajectory of a walker starting from 0 and taking a sequence of independent jumps in a random (positive or negative) direction. An interesting question concerns the rate at which the random walk can drift away from its starting point. By the SLLN, it follows that almost surely, $S_n = o(n)$, so the distance of the random walk from the origin almost surely has sub-linear growth. By the exercise above, the stronger result $S_n = o(n^{1/2+\epsilon})$ also holds for all ϵ. This is close to optimal, since by the Central Limit Theorem which we will discuss soon, one cannot hope to show that $S_n = o(n^{1/2})$. In fact, the "true" rate of growth is given by the following famous theorem, whose proof is a (somewhat complicated) elaboration on the techniques we have discussed.

Theorem (The Law of the Iterated Logarithm (A. Y. Khinchin, 1924))**.**

$$\mathbf{P}\left(\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log n}} = 1\right) = 1.$$

Therefore, by symmetry, also

$$\mathbf{P}\left(\liminf_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log n}} = -1\right) = 1.$$

It follows in particular that, almost surely, the random walk will cross the origin infinitely many times.

2. Prove that if F and $(F_n)_{n=1}^{\infty}$ are distribution functions, F is continuous, and $F_n(t) \to F(t)$ as $n \to \infty$ for any $t \in \mathbb{R}$, then the convergence is uniform in t.

3. Let $\varphi(x) = (2\pi)^{-1/2} e^{-x^2/2}$ be the standard normal density function.

(a) If $X_1, X_2, \ldots$ are i.i.d. Poisson(1) random variables and $S_n = \sum_{k=1}^{n} X_k$ (so $S_n \sim$ Poisson(n)), show that if n is large and k is an integer such that $k \approx n + x\sqrt{n}$ then

$$\mathbf{P}(S_n = k) \approx \frac{1}{\sqrt{n}} \varphi(x).$$

Hint: Use the fact that $\log(1+u) = u - u^2/2 + O(u^3)$ as $u \to 0$.

(b) Find $\lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!}$.

(c) If $X_1, X_2, \ldots$ are i.i.d. Exp(1) random variables and denote $S_n = \sum_{k=1}^{n} X_k$ (so $S_n \sim$ Gamma(n, 1)), $\hat{S}_n = (S_n - n)/\sqrt{n}$. Show that if n is large and $x \in \mathbb{R}$ is fixed then the density of $\hat{S}_n$ satisfies

$$f_{\hat{S}_n}(x) \approx \varphi(x).$$

4. Prove that if $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$ are independent r.v.'s, then $X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

Hint: First, show why it is enough to prove the following statement: If $U, V \sim N(0,1)$ are independent and $a^2 + b^2 = 1$, then $W = aU + bV \sim N(0,1)$. Then, to prove this, introduce another auxiliary variable $Z = -bU + aV$, and consider the two-dimensional transformation $(U, V) \to (W, Z)$. Apply the formula

$$f_{\phi(U,V)}(w,z) = \frac{1}{|J_\phi(\phi^{-1}(w,z))|} f_{U,V}(\phi^{-1}(w,z))$$

for the density of a transformed random 2-d vector to get the joint density of W, Z.