Homework Set No. 8 – Probability Theory (235A), Fall 2009

Posted: 11/17/09 — Due: 11/24/09

1. (a) Prove that if $X, (X_n)_{n=1}^\infty$ are random variables such that $X_n \to X$ in probability then $X_n \implies X$.

(b) Prove that if $X_n \implies c$ where $c \in \mathbb{R}$ is a constant, then $X_n \to c$ in probability.

(c) Prove that if $Z, (X_n)_{n=1}^\infty, (Y_n)_{n=1}^\infty$ are random variables such that $X_n \implies Z$ and $X_n - Y_n \to 0$ in probability, then $Y_n \implies Z$.

2. (a) Let $X, (X_n)_{n=1}^\infty$ be integer-valued r.v.'s. Show that $X_n \implies X$ if and only if $\mathbf{P}(X_n = k) \to \mathbf{P}(X = k)$ for any $k \in \mathbb{Z}$.

(b) If $\lambda > 0$ is a fixed number, and for each n, Z_n is a r.v. with distribution Binomial$(n, \lambda/n)$, show that

$$Z_n \implies \text{Poisson}(\lambda).$$

3. Let $f(x) = (2\pi)^{-1/2} e^{-x^2/2}$ be the density function of the standard normal distribution, and let $\Phi(x) = \int_{-\infty}^x f(u)\,du$ be its c.d.f. Prove the inequalities

$$\frac{1}{x + x^{-1}} f(x) \le 1 - \Phi(x) \le \frac{1}{x} f(x), \qquad (x > 0). \tag{1}$$

Note that for large x this gives a very accurate two-sided bound for the tail of the normal distribution. In fact, it can be shown that

$$1 - \Phi(x) = f(x) \cdot \cfrac{1}{x + \cfrac{1}{x + \cfrac{2}{x + \cfrac{3}{x + \cfrac{4}{x + \ldots}}}}}$$

which gives a relatively efficient method of estimating $\Phi(x)$.

Hint: To prove the upper bound in (1), use the fact that for $t > x$ we have $e^{-t^2/2} \le (t/x) e^{-t^2/2}$. For the lower bound, use the identity

$$\frac{d}{dx}\left(\frac{e^{-x^2/2}}{x}\right) = -\left(1 + \frac{1}{x^2}\right) e^{-x^2/2}$$

to compute $\int_x^\infty (1 + u^{-2}) e^{-u^2/2}\,du$. On the other hand, show that this integral is bounded from above by $(1 + x^{-2}) \int_x^\infty e^{-u^2/2}\,du$.

4. **(a)** Let $X_1, X_2, \ldots$ be a sequence of independent r.v.'s that are uniformly distributed on $\{1, \ldots, n\}$. Define

$$T_n = \min\{k : X_k = X_m \text{ for some } m < k\}.$$

If the X_j's represent the birthdays of some sequence of people on a planet in which the calendar year has n days, then T_n represents the number of people in the list who have to declare their birthdays before two people are found to have the same birthday. Show that

$$\mathbf{P}(T_n > k) = \prod_{m=1}^{k-1} \left(1 - \frac{m}{n}\right), \qquad (k \geq 2),$$

and use this to prove that

$$\frac{T_n}{\sqrt{n}} \implies F_{\text{birthday}},$$

where F_{birthday} is the distribution function defined by

$$F_{\text{birthday}}(x) = \begin{cases} 0 & x < 0, \\ 1 - e^{-x^2/2} & x \geq 0 \end{cases}$$

(note: this is not the same as the normal distribution!)

(b) Take $n = 365$. Assuming that the approximation $F_{T_n/\sqrt{n}} \approx F_{\text{birthday}}$ is good for such a value of n, estimate what is the minimal number of students that have to be put into a classroom so that the probability that two of them have the same birthday exceeds 50%. (Ignore leap years, and assume for simplicity that birthdays are distributed uniformly throughout the year; in practice this is not entirely true.)

5. Consider the following two-step experiment: First, we choose a uniform random variable $U \sim U(0, 1)$. Then, conditioned on the event $U = u$, we perform a sequence of n coin tosses with bias u, i.e., we have a sequence $X_1, X_2, \ldots, X_n$ such that conditioned on the event $U = u$, the X_k's are independent and have distribution $\text{Binom}(1, u)$. (Note: without this conditioning, the X_k's are not independent!)

Let $S_n = \sum_{k=1}^{n} X_k$. Assume that we know that $S_n = k$, but don't know the value of U. What is our subjective estimate of the probability distribution of U given this information? Show that the conditional distribution of U given that $S_n = k$ is the beta distribution $\text{Beta}(k+1, n-k+1)$. In other words, show that

$$\mathbf{P}(U \le x \mid S_n = k) = \frac{1}{B(k, n-k)} \int_0^x u^k (1-u)^{n-k} \, du, \qquad (0 \le x \le 1).$$

Note: This problem has been whimsically suggested by Laplace in the 18th century as a way to estimate the probability that the sun will rise tomorrow, given the knowledge that it has risen in the last n days. (Of course, this assumes the unlikely theological scenario whereby at the dawn of history, a $U(0,1)$ random number U was drawn, and that subsequently, every day an independent experiment was performed with probability U of success, such that if the experiment is successful then the sun rises.)

Hint: Use the following density version of the total probability formula: If A is an event and X is a random variable with density f_X, then

$$\mathbf{P}(A) = \int_{\mathbb{R}} f_X(u) \mathbf{P}(A \mid X = u) \, du.$$

Note that we have not defined what it means to condition on a 0-probability event (this is a somewhat delicate subject that we will not discuss in this quarter) - but don't worry about it, it is possible to use the formula in computations anyway and get results.