Homework Set No. 9 – Probability Theory (235A), Fall 2009

Posted: 11/24/09 — Due: Friday, 12/4/09 (Note extended due date!)

1. Compute the characteristic functions for the following distributions.

(a) **Poisson distribution:** $X \sim \text{Poisson}(\lambda)$.

(b) **Geometric distribution:** $X \sim \text{Geom}(p)$ (assume a geometric that starts at 1).

(c) **Uniform distribution:** $X \sim U[a, b]$, and in particular $X \sim [-1, 1]$ which is especially symmetric and useful in applications.

(d) **Exponential distribution:** $X \sim \text{Exp}(\lambda)$.

(e) **Symmetrized exponential:** A r.v. Z with density function $f_Z(x) = \frac{1}{2}e^{-|x|}$. Note that this is the distribution of the exponential distribution after being "symmetrized" in either of two ways: (i) We showed that if $X, Y \sim \text{Exp}(1)$ are independent then $X - Y$ has density $\frac{1}{2}e^{-|x|}$; (ii) alternatively, it is the distribution of an "exponential variable with random sign", namely $\varepsilon \cdot X$ where $X \sim \text{Exp}(1)$ and ε is a random sign (same as the coin flip distribution mentioned above) that is independent of X.

2. **(a)** If X is a r.v., show that $\text{Re}(\varphi_X)$ (the real part of φ_X) and $|\varphi_X|^2 = \varphi_X \overline{\varphi_X}$ are also characteristic functions (i.e., construct r.v.'s Y and Z such that $\varphi_Y(t) = \text{Re}(\varphi_X(t))$, $\varphi_Z(t) = |\varphi_X(t)|^2$).

(b) Show that X is equal in distribution to $-X$ if and only if φ_X is a real-valued function.

3. **(a)** Let $Z_1, Z_2, \ldots$ be a sequence of independent r.v.'s such that the random series $X = \sum_{n=1}^{\infty} Z_n$ converges a.s. Prove that

$$\varphi_X(t) = \prod_{n=1}^{\infty} \varphi_{Z_n}(t), \qquad (t \in \mathbb{R}).$$

(b) Let X be a uniform r.v. in $(0, 1)$, and let $Y_1, Y_2, \ldots$ be the (random) bits in its binary expansion, i.e. each Y_n is either 0 or 1, and the equation

$$X = \sum_{n=1}^{\infty} \frac{Y_n}{2^n} \tag{1}$$

holds. Show that $Y_1, Y_2, \ldots$ are i.i.d. unbiased coin tosses (i.e., taking values $0, 1$ with probabilities $1/2, 1/2$).

(c) Compute the characteristic function φ_Z of $Z = 2X - 1$ (which is uniform in $(-1, 1)$). Use (1) to represent this in terms of the characteristic functions of the Y_n's (note that the series (1) converges absolutely, so here there is no need to worry about almost sure convergence). Deduce the infinite product identity

$$\frac{\sin(t)}{t} = \prod_{n=1}^{\infty} \cos\left(\frac{t}{2^n}\right), \qquad (t \in \mathbb{R}). \tag{2}$$

(d) Substitute $t = \pi/2$ in (2) to get the identity

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdot \ldots$$

4. Let X be a r.v. From the inversion formula, it follows without much difficulty (see Theorem (3.3), p. 95 in [Durrett], 3rd ed.), that if φ_X is integrable, then X has a density f_X, and the density and characteristic function are related by

$$\begin{aligned} \varphi_X(t) &= \int_{-\infty}^{\infty} f_X(x) e^{itx}\, dx, \\ f_X(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi_X(t) e^{-itx}\, dt \end{aligned}$$

(this shows the duality between the Fourier transform and its inverse). Use this and the answer to question 1(e) above to conclude that if X is a r.v. with the Cauchy distribution (i.e., X has density $f_X(x) = 1/\pi(1 + x^2)$) then its characteristic function is given by

$$\varphi_X(t) = e^{-|t|}.$$

Deduce from this that if X, Y are independent Cauchy r.v.'s then any weighted average $\lambda X + (1 - \lambda)Y$, where $0 \le \lambda \le 1$, is also a Cauchy r.v. (As a special case, it follows by induction that if $X_1, \ldots, X_n$ are i.i.d. Cauchy r.v.'s, then their average $(X_1 + \ldots + X_n)/n$ is also a Cauchy r.v., which was a claim we made without proof earlier in the course.)

Happy Thanksgiving!