Question 1 (25 points)

Use matrix exponentials, or any other method, to find the solution $(x(t), y(t))$ of the ODE system

$$\dot{x} = 2x - y,$$
$$\dot{y} = 2x - y,$$

satisfying the initial conditions $x(0) = 0, y(0) = 1$.

Solution. In matrix notation, the system becomes $\dot{\mathbf{x}} = A\mathbf{x}$, where $A = \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix}$. A quick computation shows that A has eigenvalues $\lambda_1 = 0, \lambda_2 = 1$ with associated eigenvectors $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. That means that A can be expressed as $A = PDP^{-1}$ where

$$P = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Denote the initial condition vector by $\mathbf{x}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The solution to the ODE system is given by

$$\begin{aligned}
\mathbf{x}(t) &= e^{tA}\mathbf{x}_0 = e^{tPDP^{-1}}\mathbf{x}_0 = Pe^{tD}P^{-1}\mathbf{x}_0 \\
&= \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^t \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & e^t \\ 2 & e^t \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 + 2e^t & 1 - e^t \\ -2 + 2e^t & 2 - e^t \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 - e^t \\ 2 - e^t \end{pmatrix}.
\end{aligned}$$

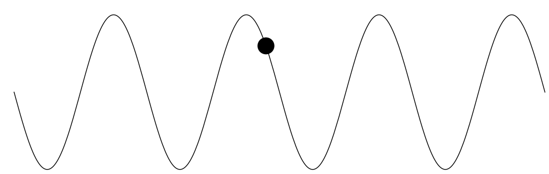
To summarize, the required solution is

$$x(t) = 1 - e^t, \quad y(t) = 2 - e^t.$$

Question 2 (25 points)

A roller-coaster rolls under the influence of gravity along a track shaped like the graph of the function $z = \sin x$ (where z is the vertical coordinate, x is the horizontal coordinate along which the motion takes place, and there is no motion along the y-axis; see the figure below). Let $q = q(t)$ denote the x-coordinate of the roller-coaster at time t, so that its complete position in space is

$$\mathbf{x}(t) = (q, 0, \sin q).$$



(a) (7 points) Find the Lagrangian $L(\dot{q}, q)$ associated with this system.

Solution.

$$\begin{aligned} K &= \text{kinetic energy} = \tfrac{1}{2}|\dot{\mathbf{x}}|^2 = \tfrac{1}{2}(\dot{q}^2 + (\dot{q}\cos q)^2) = \tfrac{1}{2}\dot{q}^2(1+\cos^2 q), \\ U &= \text{potential energy} = g\sin q, \\ L &= K - U = \tfrac{1}{2}\dot{q}^2(1+\cos^2 q) - g\sin q. \end{aligned}$$

(b) (7 points) Find the associated Hamiltonian (energy function) $H(p, q)$. Note that H must be expressed as a function of the variables $p = \frac{\partial L}{\partial \dot{q}}$ and q, not $\dot{q}$ and q.

Solution. The generalized force p is $p = \frac{\partial L}{\partial \dot{q}} = \dot{q}(1+\cos^2 q)$. The Hamiltonian is

$$\begin{aligned} H(p,q) &= p\dot{q} - L = p\frac{p}{1+\cos^2 q} + g\sin q - \tfrac{1}{2}\left(\frac{p}{1+\cos^2 q}\right)^2(1+\cos^2 q) \\ &= \tfrac{1}{2}\frac{p^2}{1+\cos^2 q} + g\sin q. \end{aligned}$$

(It is also possible to arrive at the same result using the formula $H = K + U$ which is valid in a conservative system).

(c) (6 points) From the Hamiltonian H you found, derive Hamilton's equations for the system.

Solution. The partial derivatives of H are

$$\begin{aligned} \frac{\partial H}{\partial p} &= \frac{p}{1+\cos^2 q}, \\ \frac{\partial H}{\partial q} &= g\cos q + \frac{p^2\cos q\sin q}{(1+\cos^2 q)^2} = \cos q\left(g + \frac{p^2 \sin q}{(1+\cos^2 q)^2}\right), \end{aligned}$$

so Hamilton's equations are

$$\dot{p} = -\frac{\partial H}{\partial q} = -\cos q \left(g + \frac{p^2 \sin q}{(1+\cos^2 q)^2} \right),$$
$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{1+\cos^2 q}.$$

(d) (5 points) Identify all the rest points (p, q) of the system in Hamiltonian form.

Solution. At a rest point we have $\dot{p} = \dot{q} = 0$ so from the equations above the condition for a rest point is

$$p = 0, \quad \cos q = 0.$$

The solutions are

$$(p, q) = \left(0, (n + \tfrac{1}{2})\pi\right), \qquad n = \ldots, -2, -1, 0, 1, 2, \ldots$$

Physically, they correspond to the roller-coaster being balanced at rest at one of the (stable) local minimum points or (unstable) local maximum points of the sine-shaped track.

Question 3 (25 points)

Define a family of interval maps $T_r : [0, 1] \to [0, 1]$, where $0 \le r \le 4$ is a parameter, by

$$T_r(x) = 1 - rx(1 - x).$$

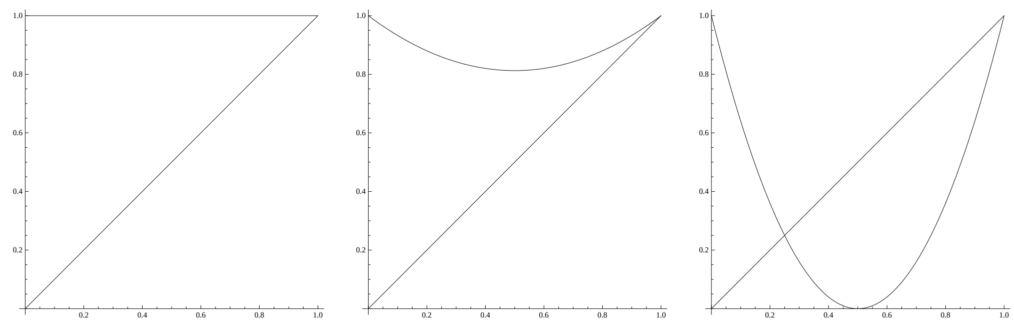
(a) (5 points) Sketch a rough picture of the graph of $T_r(x)$, along with the identity function $y = x$, for the three values of r:

i. $r = 0$

ii. $r = 4$

iii. Any value $0 < r < 4$ of your choice.

Solution.



$r = 0$ $r = 0.75$ $r = 4$

(b) (12 points) For each r, determine all the fixed points of T_r and for each fixed point use the first derivative test, if it is applicable, to determine whether it is asymptotically stable or asymptotically unstable (in this problem there are no other types of behavior).

Solution. The equation for a fixed point is $T_r(x) = x$,

$$0 = T_r(x) - x = 1 - rx(1 - x) - x = rx^2 - (r + 1)x + 1.$$

This is a quadratic equation in x. Its solutions are $x = 1$ and $x = 1/r$ (which is outside the interval $[0, 1]$ if $r < 1$). That means that for $r \le 1$ there is only one fixed point $x_* = 1$ and for $1 \le r \le 4$ there are two fixed points $x_* = 1$ and $x_* = 1/r$.

To investigate the stability, look at the derivative $\lambda = T'(x_*) = -r(1-2x)$. For $x_* = 1$ this gives $\lambda = r$, so the fixed point is asymptotically stable for $r < 1$ (since $|\lambda| < 1$) and asymptotically unstable for $r > 1$ (since $|\lambda| > 1$). (For $r = 1$ it can be seen to be asymptotically stable, for example by drawing a cobweb diagram.) Finally, for the fixed point $x_* = 1/r$ in the case $r > 1$ we have

$$\lambda = -r(1 - 2/r) = 2 - r,$$

which shows that this fixed point is asymptotically stable if $1 < r < 3$, and asymptotically unstable for $3 < r < 4$. (Again, in the boundary case $r = 3$ it can be checked that the fixed point is asymptotically stable.)

(c) (8 points) Find a value of r for which $x = \frac{1}{2}$ is part of a 2-cycle (x, y) of T_r, if such a value exists. (Note that in this case $x = \frac{1}{2}$ should *not* be a fixed point.)

Solution. The condition for $x = \frac{1}{2}$ to belong to a 2-cycle is that $T_r^2(\frac{1}{2}) = T_r(T_r(\frac{1}{2})) = \frac{1}{2}$ and $\frac{1}{2}$ should not be a fixed point, which rules out the value $r = 2$. A computation gives

$$\begin{aligned} T_r(\tfrac{1}{2}) &= 1 - r/4, \\ T_r(T_r(\tfrac{1}{2})) &= T_r(1 - r/4) = 1 - r(1 - r/4)(r/4) = 1 - \frac{r^2}{4} + \frac{r^3}{16}, \end{aligned}$$

so we get an equation for r:

$$0 = T_r(T_r(\tfrac{1}{2})) - \tfrac{1}{2} = \tfrac{1}{2} - \frac{r^2}{4} + \frac{r^3}{16} = \frac{r^3 - 4r^2 + 8}{16}.$$

This is a cubic equation. One obvious root is the value $r = 2$ already mentioned for which $x = 1/2$ is a fixed point, so we can extract a factor $(r - 2)$ to get the equation

$$r^3 - 4r^2 + 8 = (r - 2)(r^2 - 2r - 4) = 0.$$

The quadratic equation $r^2 - 2r - 4 = 0$ has roots $1 \pm \sqrt{5}$. Of these, only the root $1 + \sqrt{5} \approx 3.236$ is in the range $[0, 4]$. To conclude, the answer is $r = 1 + \sqrt{5}$.

Question 4 (25 points)

(a) (5 points) Show that the solutions $(x(t), y(t))$ to the planar system

$$\dot{x} = y,$$
$$\dot{y} = -1$$

satisfy the relationship $x = a - \frac{1}{2}y^2$ for some real number $a \in \mathbb{R}$.

Solution. Integrating the second equation gives $y(t) = -t + c$ where c is a constant of integration. Plugging this into the first equation and integrating again gives $x(t) = -\frac{1}{2}t^2 + ct + d$ where d is another constant of integration. Substituting $t = c - y$ into the formula for x gives

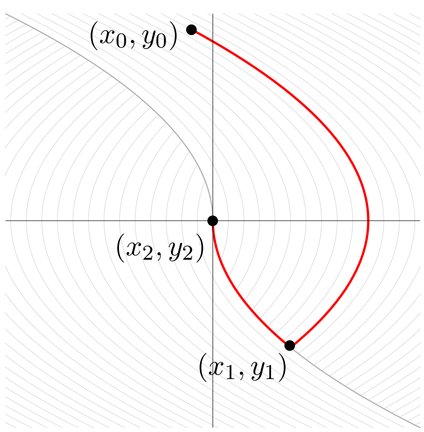
$$\begin{aligned} x &= -\tfrac{1}{2}(c-y)^2 + c(c-y) + d = -\tfrac{1}{2}y^2 + cy - cy - \tfrac{1}{2}c^2 + c^2 + d \\ &= -\tfrac{1}{2}y^2 + (d - \tfrac{1}{2}c^2). \end{aligned}$$

Denoting $a = d - \frac{1}{2}c^2$, this shows that $x = a - \frac{1}{2}y^2$, as claimed.

(b) (20 points) The optimal switching control rule in the electromagnetic levitation problem leads to the system

$$\dot{x} = y,$$
$$\dot{y} = -\operatorname{sgn}\left(x + \tfrac{1}{2}y|y|\right).$$

Find a formula for the time $\tau(x_0, y_0)$ it takes the system to get to the rest point at the origin from an arbitrary initial state (x_0, y_0), assuming that $x_0 > -\frac{1}{2}y_0|y_0|$ (i.e., that the initial state is to the right of the switching curve—see the figure below).



Solution. The point first flows along the parabola $x = a - \frac{1}{2}y^2$ (where a is determined by the condition $x_0 = a - \frac{1}{2}y_0^2$, giving $a = x_0 + \frac{1}{2}y_0^2$), until it meets the point (x_1, y_1) at the intersection of this parabola with the second parabola $x = \frac{1}{2}y^2$ (more precisely, the half-parabola where $y < 0$—see the figure). So, (x_1, y_1) satisfies the simultaneous equations

$$x_1 = a - \tfrac{1}{2}y_1^2 = \tfrac{1}{2}y_1^2,$$

from which it is easy to find that

$$(x_1, y_1) = \left(\tfrac{1}{2}(x_0 + \tfrac{1}{2}y_0^2), -\sqrt{x_0 + \tfrac{1}{2}y_0^2} \right).$$

From there, the point flows directly to $(x_2, y_2) = (0, 0)$. The total time to get to $(0, 0)$ is given by the sum of the absolute values of the differences of the y-coordinates (since, from the equations of motion, $|\dot{y}| = 1$ always):

$$\tau(x_0, y_0) = |y_0 - y_1| + |y_1 - y_2| = y_0 - 2y_1 = y_0 + 2\sqrt{x_0 + \tfrac{1}{2}y_0^2}.$$