1. (a) Starting from the Hamiltonian $H(p,q,t) = p^2 + tq^4$, find the associated Lagrangian, and write the associated ODEs in both the Lagrangian and Hamiltonian forms.

Solution. The system in Hamiltonian form: $\dot{p} = -4tq^3,\ \dot{q} = 2p$. The Lagrangian is

$$L(\dot{q}, q, t) = p\dot{q} - H(p,q,t) = \tfrac{1}{2}\dot{q}^2 - ((\tfrac{1}{2}\dot{q})^2 + tq^4) = \tfrac{1}{4}\dot{q}^2 - tq^4,$$

and the Euler-Lagrange equation is $\frac{1}{2}\ddot{q} = -4tq^3$.

(b) Starting from the Lagrangian $L(\dot{q}, q, t) = \sqrt{1+\dot{q}^2}$, find the associated Hamiltonian, and write the associated ODEs in both the Lagrangian and Hamiltonian forms.

Solution. The generalized momentum is $p = \frac{\partial L}{\partial \dot{q}} = \frac{\dot{q}}{\sqrt{1+\dot{q}^2}}$. Solving this equation for $\dot{q}$ gives $\dot{q} = \frac{p}{\sqrt{1-p^2}}$, so L expressed in terms of p is $L = \frac{1}{\sqrt{1-p^2}}$. This gives the Hamiltonian

$$H(p,q) = p\dot{q} - L = \frac{p^2}{\sqrt{1-p^2}} - \frac{1}{\sqrt{1-p^2}} = -\sqrt{1-p^2}.$$

Since $\frac{\partial L}{\partial q} = 0$, the Euler-Lagrange equation is

$$\frac{dp}{dt} = \frac{\ddot{q}\sqrt{1+\dot{q}^2} - \dot{q}^2/\sqrt{1+\dot{q}^2}}{1+\dot{q}^2} = 0.$$

The Hamiltonian form of the system is

$$\dot{p} = -\frac{\partial H}{\partial q} = 0, \quad \dot{q} = \frac{\partial H}{\partial p} = \frac{p}{\sqrt{1-p^2}}.$$

2. For each of the following planar systems, decide if the system is Hamiltonian, and if it is, find the Hamiltonian function $H(p,q,t)$:

(a) $\dot{p} = p^2 + q^2,\ \dot{q} = -2pq$

Solution. We know that the system $\dot{p} = F(p,q,t),\ \dot{q} = G(p,q,t)$ is Hamiltonian if and only if $\frac{\partial F}{\partial p} + \frac{\partial G}{\partial q} = 0$. Here $F(p,q,t) = p^2 + q^2, G(p,q,t) = -2pq$, and the condition is satisfied. To find the Hamiltonian H such that $F = -\frac{\partial H}{\partial q}$, $G = \frac{\partial H}{\partial p}$, we compute a line integral of the vector field $(-G, F)$ along an arbitrary curve from some fixed point (p_0, q_0) to (p,q). A convenient choice is the curve that goes in two straight line segments from (p_0, q_0) to (p, q_0) and then to (p,q). This gives:

$$\begin{aligned} H(p,q) &= \int_{(p_0,q_0)}^{(p,q)} G\,dp - F\,dq = \int_{p_0}^{p} G(u,q_0)\,du - \int_{q_0}^{q} F(p,v)\,dv \\ &= \int_{p_0}^{p} -2uq_0\,du - \int_{q_0}^{q} (p^2+v^2)\,dv = -q_0(p^2 - p_0^2) - p^2(q-q_0) - \tfrac{1}{3}(q^3 - q_0^3) \\ &= -p^2 q - \tfrac{1}{3}q^3 + \text{const}. \end{aligned}$$

(b) $\dot{p} = e^t p,\ \dot{q} = 10t$

Solution. The condition $\frac{\partial}{\partial p}\left(e^t p\right) + \frac{\partial}{\partial q}(10t) = 0$ is not satisfied, so the system is not Hamiltonian.

(c) $\dot{p} = e^p \sin q,\ \dot{q} = e^p \cos q$

Solution. The system is Hamiltonian. The Hamiltonian is $H(p,q) = e^p \cos q$.

3. Given a smooth function $f(x)$ defined on some interval $[a,b]$, and assuming that f is strictly convex (i.e., $f'' > 0$), its *Legendre transform* is a function $g(p)$ defined on the interval $[c,d]$, where $c = f'(a), d = f'(b)$. To compute $g(p)$, first find the point x such that $p = f'(x)$, and then set

$$g(p) = px - f(x).$$

(a) Prove that the Legendre transform is its own inverse: i.e., $f(x)$ is the Legendre transform of $g(p)$.

Solution. The Legendre transform of g can be written as the function $h(y)$ satisfying

$$h(y) = yp - g(p),$$

where y and p are related via $g'(p) = y$. If we express $h(y)$ in terms of the variables p and x, we have

$$h(y) = g'(p)p - g(p) = pg'(p) - px + f(y) = p(g'(p) - x) + f(x).$$

But note that

$$g'(p) = \frac{dg}{dp} = \frac{d}{dp}(px - f(x)) = x + p\frac{dx}{dp} - f'(x)\frac{dx}{dp} = x + p\frac{dx}{dp} - p\frac{dx}{dp} = x,$$

so the above equation for $h(y)$ turns into

$$h(y) = f(x),$$

and since we also have $x = g'(p) = y$, we get $h(y) = f(x) = f(y)$, i.e., $f(x)$ is the Legendre transform of its own Legendre transform $g(p)$.

(b) Prove that $g'' > 0$, i.e., g is also strictly convex.

Solution. From the relation $p = f'(x)$ we get that $f''(x)$, which is assumed to be positive, can be written as $f''(x) = \frac{dp}{dx}$. But then similarly $g''(p) = \frac{dy}{dp} = \frac{dx}{dp} = \left(\frac{dp}{dx}\right)^{-1} = f''(x)^{-1}$, so also $g''(p) > 0$.

(c) Compute the Legendre transforms of the following functions:

i. $f(x) = x^\alpha, \alpha > 1$.

Solution. $g(p) = \left(1 - \frac{1}{\alpha}\right)\left(\frac{1}{\alpha}\right)^{1/(\alpha-1)} p^{\alpha/(\alpha-1)}$.

ii. $f(x) = e^x$

Solution. $g(p) = p \log p - p$.

iii. $f(x) = \cosh x$

Solution. $g(p) = p \sinh^{-1} p - \sqrt{1+p^2}$.

4. A particle moving in the plane, whose position at time t is denoted by $\mathbf{x}(t) = (x(t), y(t))$, satisfies the equations of motion

$$\begin{aligned} \ddot{x} &= \ \ B(x,y)\dot{y}, \\ \ddot{y} &= -B(x,y)\dot{x}, \end{aligned}$$

where $B = B(x,y)$ is a sufficiently smooth, scalar, time-independent quantity. (Note that this means that the force acting on the particle is proportional in magnitude to $B(x,y)$ and to the particle's speed, and its direction is orthogonal to the velocity vector, since $\ddot{\mathbf{x}} \cdot \dot{\mathbf{x}} = 0$.)

(a) Show that if $P(x,y)$ and $Q(x,y)$ are functions such that we have the relation

$$B(x,y) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y},$$

then the system can be described in terms of the Euler-Lagrange equations associated with the Lagrangian

$$L(\dot{x}, x, \dot{y}, y) = \tfrac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) + P(x,y)\dot{x} + Q(x,y)\dot{y}.$$

Solution. Since this is a system with two degrees of freedom, we have two Euler-Lagrange equations:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) &= \frac{\partial L}{\partial x}, \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) &= \frac{\partial L}{\partial y}. \end{aligned}$$

Computing the partial derivatives of L with respect to $x, \dot{x}, y, \dot{y}$ gives:

$$\begin{aligned} \frac{\partial L}{\partial \dot{x}} &= \dot{x} + P(x,y), \\ \frac{\partial L}{\partial \dot{y}} &= \dot{y} + Q(x,y), \\ \frac{\partial L}{\partial x} &= \frac{\partial P}{\partial x}\dot{x} + \frac{\partial Q}{\partial x}\dot{y}, \\ \frac{\partial L}{\partial y} &= \frac{\partial P}{\partial y}\dot{x} + \frac{\partial Q}{\partial y}\dot{y}. \end{aligned}$$

So the Euler-Lagrange equations become

$$\ddot{x} + \frac{\partial P}{\partial x}\dot{x} + \frac{\partial P}{\partial y}\dot{y} = \ddot{x} + \frac{dP}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial p}{\partial x}\dot{x} + \frac{\partial Q}{\partial x}\dot{y},$$
$$\ddot{y} + \frac{\partial Q}{\partial x}\dot{x} + \frac{\partial Q}{\partial y}\dot{y} = \ddot{x} + \frac{dQ}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = \frac{\partial P}{\partial y}\dot{x} + \frac{\partial Q}{\partial y}\dot{y}.$$

Rearranging the terms gives

$$\ddot{x} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\dot{y} = B(x,y)\dot{y},$$
$$\ddot{y} = \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)\dot{x} = -B(x,y)\dot{x},$$

which are the equations we wanted to get.

(a) There are many possible choices of P and Q, for example choosing $P \neq 0$ will determine Q up to a constant. Can you characterize all possible pairs (P, Q) which work for a given function $B(x, y)$?

Solution. Fix a pair of functions (P_0, Q_0) for which $B(x, y) = \frac{\partial Q_0}{\partial x} - \frac{\partial P_0}{\partial y}$. If (P, Q) is any other pair of functions that satisfies the same equation, then the vector field $(F, G) = (P - P_0, Q - Q_0)$ has the property that

$$\mathrm{curl}(F, G) = \frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} = 0,$$

i.e. (F, G) is an irrotational field. From vector calculus we know that (assuming that F, G are defined on a simply connected region of the plane) this happens if and only if (F, G) is derived from a potential, i.e., if it can be represented as $(F, G) = \nabla U$ for some scalar function $U(x, y)$. It follows that the most general form for a pair (P, Q) of functions suitable for use in the above Lagrangian is

$$P(x,y) = P_0(x,y) + \frac{\partial U}{\partial x},$$
$$Q(x,y) = Q_0(x,y) + \frac{\partial U}{\partial y},$$

where $U(x, y)$ is an arbitrary (sufficiently smooth) function.

(b) Find the solution of the system with initial conditions

$$\mathbf{x}(0) = (0, 0),$$
$$\dot{\mathbf{x}}(0) = (v, 0),$$

in the case $B(x, y) \equiv b = \text{const}$. Describe the behavior of the particle in words.

Solution. The particle moves in a circle. The solution is

$$x(t) = \frac{v}{b}\sin(bt),$$
$$y(t) = \frac{v}{b}\cos(bt) - \frac{v}{b}.$$