Homework due: Friday 4/13 in class

Grading guidelines: The homework will not be graded in detail. I will mark an assignment as $0\%, 50\%$ or 100% complete. For a grade of 50%, solve at least one problem. For a grade of 100%, solve at least three problems.

Problems

1. (a) Starting from the Hamiltonian $H(p, q, t) = p^2 + tq^4$, find the associated Lagrangian, and write the associated ODEs in both the Lagrangian and Hamiltonian forms.

   (b) Starting from the Lagrangian $L(\dot{q}, q, t) = \sqrt{1 + \dot{q}^2}$, find the associated Hamiltonian, and write the associated ODEs in both the Lagrangian and Hamiltonian forms.

2. For each of the following planar systems, decide if the system is Hamiltonian, and if it is, find the Hamiltonian function $H(p, q, t)$:

   (a) $\dot{p} = p^2 + q^2,\ \dot{q} = -2pq$

   (b) $\dot{p} = e^t p,\ \dot{q} = 10t$

   (c) $\dot{p} = e^p \sin q,\ \dot{q} = e^p \cos q$

3. Given a smooth function $f(x)$ defined on some interval $[a, b]$, and assuming that f is strictly convex (i.e., $f'' > 0$), its *Legendre transform* is a function $g(p)$ defined on the interval $[c, d]$, where $c = f'(a), d = f'(b)$. To compute $g(p)$, first find the point x such that $p = f'(x)$, and then set

$$g(p) = px - f(x).$$

   (a) Prove that the Legendre transform is its own inverse: i.e., $f(x)$ is the Legendre transform of $g(p)$.

   (b) Prove that $g'' > 0$, i.e., g is also strictly convex.

   (c) Compute the Legendre transforms of the following functions:

      i. $f(x) = x^\alpha, \alpha > 1$

      ii. $f(x) = e^x$

      iii. $f(x) = \cosh x$

   (d) Convince yourself (no need to explain this in writing) that the Hamiltonian $H(p, q, t)$ is the Legendre transform of the Lagrangian $L(\dot{q}, q, t)$, when considered as a function of $\dot{q}$ for fixed q, t.

4. A particle moving in the plane, whose position at time t is denoted by $\mathbf{x}(t) = (x(t), y(t))$, satisfies the equations of motion

$$\begin{aligned} \ddot{x} &= \quad B(x, y)\dot{y}, \\ \ddot{y} &= -B(x, y)\dot{x}, \end{aligned}$$

   where $B = B(x, y)$ is a sufficiently smooth, scalar, time-independent quantity. (Note that this means that the force acting on the particle is proportional in magnitude to $B(x, y)$ and to the particle's speed, and its direction is orthogonal to the velocity vector, since $\ddot{\mathbf{x}} \cdot \dot{\mathbf{x}} = 0$.)

(a) Show that if $P(x,y)$ and $Q(x,y)$ are functions such that we have the relation

$$B(x,y) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y},$$

then the system can be described in terms of the Euler-Lagrange equations associated with the Lagrangian

$$L(\dot{x},x,\dot{y},y) = \tfrac{1}{2}\left(\dot{x}^2+\dot{y}^2\right) + P(x,y)\dot{x} + Q(x,y)\dot{y}.$$

(Note that there are many possible choices of P and Q, for example choosing $P \neq 0$ will determine Q up to a constant. Can you characterize all possible pairs (P,Q) which work for a given function $B(x,y)$?)

(b) Find the solution of the system with initial conditions

$$\mathbf{x}(0) = (0,0),$$
$$\dot{\mathbf{x}}(0) = (v,0),$$

in the case $B(x,y) \equiv b = \text{const}$. Describe the behavior of the particle in words.

(c) (Optional) The above system can be interpreted physically as a magnetic force acting on a charged particle. A more general situation involving a combination of time-dependent electric and magnetic forces on a particle in $\mathbb{R}^3$ (known as the *Lorentz force*) is described in exercise 2, pages 13–14 in the course lecture notes. For bonus points, show that writing the Euler-Lagrange equations for the full Lagrangian described in that exercise reproduces the Lorentz force equation.