1. In the computer game "Angry Birds Space," let $\mathbf{x}(t) = (x(t), y(t))$ denote the position of an angry bird at time t flying near a planet centered at $(0,0)$, and denote $r(t) = |\mathbf{x}(t)| = \sqrt{x(t)^2 + y(t)^2}$. Experimental evidence suggests[1] that $\mathbf{x}$ satisfies the equation

$$\ddot{\mathbf{x}} = -F(r)\frac{\mathbf{x}}{r}$$

for some unknown function $F(r)$ (since $\mathbf{x}/r$ is a unit vector, $F(r)$ represents the force of attraction towards the planet). The force function $F(\cdot)$ is a function of a single variable, so we can associate with it a potential function $U(r)$ such that $F(r) = -U'(r)$.

(a) Show that the quantity $M = x\dot{y} - y\dot{x}$ (the "angry bird angular momentum") is conserved along trajectories, i.e., $\frac{dM}{dt} = 0$.

Solution.

$$\frac{dM}{dt} = \frac{d}{dt}(x\dot{y} - y\dot{x}) = \dot{x}\dot{y} + x\ddot{y} - \dot{y}\dot{x} - y\ddot{x} = -U'(r)\left(x\frac{y}{r} - y\frac{x}{r}\right) = 0.$$

(b) Show that the radial distance $r(t)$ of the angry bird from the center of the planet satisfies the equation

$$\ddot{r} = -U'(r) + \frac{M^2}{r^3},$$

where M is the constant of motion from part (a) above (note that M is a function of the initial conditions $\mathbf{x}(0), \dot{\mathbf{x}}(0)$). An equivalent way of saying this is that r is derived from a conservative system with one degree of freedom $\ddot{r} = -V'(r)$, with effective potential $V(r) = U(r) + \frac{M^2}{2r^2}$.

Solution. Write the system in components:

$$\ddot{x} = -U'(r)\frac{x}{r}, \quad \ddot{y} = -U'(r)\frac{y}{r},$$

then differentiate r twice, to get:

$$\begin{aligned}
\dot{r} &= \frac{d}{dt}\sqrt{x^2+y^2} = \frac{x}{\sqrt{x^2+y^2}}\dot{x} + \frac{y}{\sqrt{x^2+y^2}}\dot{y} = \frac{x\dot{x}+y\dot{y}}{r}, \\
\ddot{r} &= \frac{\left(x\ddot{x} + y\ddot{y} + \dot{x}^2 + \dot{y}^2\right)r - (x\dot{x}+y\dot{y})\,\dot{r}}{r^2} \\
&= -U'(r)\frac{x\cdot\frac{x}{r} + y\cdot\frac{y}{r}}{r} + \frac{\left(\dot{x}^2+\dot{y}^2\right)r - (x\dot{x}+y\dot{y})^2/r}{r^2} \\
&= -U'(r) + \frac{(\dot{x}^2+\dot{y}^2)(x^2+y^2) - (x\dot{x}+y\dot{y})^2}{r^3} \\
&= -U'(r) + \frac{(x\dot{y}-y\dot{x})^2}{r^3} = -U'(r) + \frac{M^2}{r^3} = -\frac{d}{dt}\left(U(r) + \frac{M^2}{2r^2}\right).
\end{aligned}$$

[1] see: `http://www.wired.com/wiredscience/2012/03/the-gravitational-force-in-angry-birds-space`

(c) Let $U(r) = -\frac{k}{r}$, where $k > 0$ (corresponding to the case of ordinary Newtonian gravitation). Find the minimal and maximal radial distances r_{min} and r_{max} as a function of k and of the initial conditions $\mathbf{x}(0), \dot{\mathbf{x}}(0)$. This should include a condition for when $r_{\text{max}} = \infty$, i.e., a characterization of when the motion is unbounded.

Hint. Use parts (a) and (b) above as well as the fact that $E = \frac{1}{2}\dot{r}^2 + V(r)$ is a conserved quantity. Try to express $r_{\text{min}}, r_{\text{max}}$ in terms of E and M where possible. You will need to divide into several cases.

Solution. The energy associated with the radial motion,

$$E = \tfrac{1}{2}\dot{r}^2 + V(r),$$

is a conserved quantity. When r is at an extremum (minimum or maximum) point, we have $\dot{r} = 0$, that is, $V(r) = E$ (equivalently, this happens when the kinetic energy component, $\frac{1}{2}\dot{r}^2$ is zero, which means all the energy is in potential form). So we get the equation

$$V(r) = -\frac{k}{r} + \frac{M^2}{2r^2} = E,$$

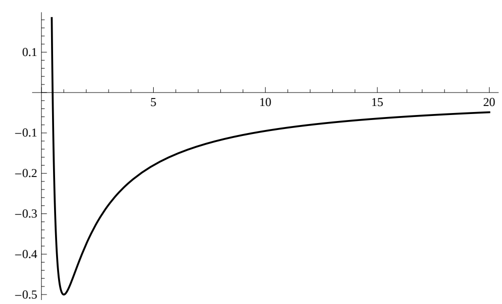
or

$$2Er^2 + 2kr - M^2 = 0.$$

Solving for r gives

$$r_{1,2} = \frac{-k \pm \sqrt{k^2 + 2M^2E}}{2E}.$$

We now need to divide into cases according to whether the two solutions correspond to physical states $r > 0$. It is helpful to look at a picture of the potential $V(r)$, which looks something like this:



We see that when $E \geq 0$, the equation $V(r) = E$ has only one physical solution $r > 0$. In this case, the motion is unbounded and $r_{\text{max}} = \infty$. The value of r_{min} will depend on whether the initial conditions lead to the radius initially decreasing (in which case it will decrease all the way to the positive solution $r_1 = \frac{-k+\sqrt{k^2+2M^2E}}{2E}$ before changing direction) or initially increasing (in which case it will continue increasing forever, so $r_{\text{min}} = r(0)$).

For the case $E < 0$, the two solutions $r_{1,2}$ are both either positive (for $E \geq -k^2/2M^2$) or are not real numbers (for $E < -k^2/2M^2$). The latter case is impossible, since it means the initial radial kinetic energy term $\frac{1}{2}\dot{r}^2$ would be negative. When the two solutions are real, that means the motion is bounded, with the extremal radial values

$$r_{\min} = \frac{-k + \sqrt{k^2 + 2M^2E}}{2E}, \qquad r_{\max} = \frac{-k - \sqrt{k^2 + 2M^2E}}{2E}.$$

To summarize, the answer is expressed in three cases:

$$(r_{\min}, r_{\max}) = \begin{cases} \left(\frac{-k+\sqrt{k^2+2M^2E}}{2E}, \infty\right) & \text{if } E \geq 0, \dot{r}(0) < 0, \\ (r(0), \infty) & \text{if } E \geq 0, \dot{r} \geq 0, \\ \left(\frac{-k+\sqrt{k^2+2M^2E}}{2E}, \frac{-k-\sqrt{k^2+2M^2E}}{2E}\right) & \text{if } -\frac{k^2}{2M^2} \leq E < 0. \end{cases}$$

Note that in the case $E = -k^2/2M^2$ we have $r_{\min} = r_{\max}$. In this case r is a constant and the orbit is a circle.

(d) Repeat part (c) for a potential of the form $U(r) = \frac{1}{2}kr^2$ $(k > 0)$, corresponding to a two-dimensional harmonic oscillator.

Solution. In this case the motion is always bounded, since the effective potential $V(r) = \frac{1}{2}kr^2 + \frac{M^2}{2r^2}$ is unbounded as $r \to \infty$. The values of $r_{\min,\max}$ are the solutions of the equation $V(r) = E$, i.e., $\frac{1}{2}kr^2 + \frac{M^2}{2r^2} = E$, or equivalently

$$kr^4 - 2Er^2 + M^2 = 0$$

(a quadratic equation in r^2). The solutions are

$$r_{\min} = \sqrt{\frac{E - \sqrt{E^2 - M^2k}}{k}}, \quad r_{\max} = \sqrt{\frac{E + \sqrt{E^2 - M^2k}}{k}}.$$

2. The *Lotke-Volterra predator-prey equations* are given by

$$\begin{aligned} \dot{x} &= x(a - cy), \\ \dot{y} &= y(-b + dx), \end{aligned} \qquad (x, y > 0),$$

where a, b, c, d are positive parameters. This planar system of ODEs is used to model the interaction over time of the population sizes (represented by the dynamic variables x, y) of two biological species with one population preying on the other.

(a) Show that the change of variables $p = \log x$, $q = \log y$ transforms the system into a Hamiltonian system $\dot{p} = -\frac{\partial H}{\partial q}, \dot{q} = \frac{\partial H}{\partial p}$, and find the associated Hamiltonian.

Solution. Differentiating p and q gives

$$\dot{p} = \frac{\dot{x}}{x} = a - cy = a - ce^q,$$
$$\dot{q} = \frac{\dot{y}}{y} = -b + dx = -b + de^p.$$

The vector field $(a - ce^q, -b + de^p)$ has zero divergence, so the system is Hamiltonian. The Hamiltonian is found by computing a line integral of the vector field with one endpoint fixed. The result is

$$H(p, q) = -aq + ce^q - bp + de^p.$$

(b) Use the fact that the Hamiltonian you found is autonomous to write down a conserved quantity $G(x, y)$ (of the original system). Note that this gives a family of implicit equations $G(x, y) = C$ which for various values of the constant C describe the shape of the solution curves in the x-y plane.

Solution. The Hamiltonian $H(p, q)$ is a conserved quantity, so it suffices to express it as a function of x and y:

$$G(x, y) = H(p, q) = H(\log x, \log y) = -a \log y + cy - b \log x + dx.$$

Verification that G is conserved:

$$\begin{aligned}\dot{G} &= \frac{\partial G}{\partial x}\dot{x} + \frac{\partial G}{\partial y}\dot{y} = \left(d - \frac{b}{x}\right) x(a - cy) + \left(c - \frac{a}{y}\right) y(-b + dx) \\ &= (dx - b)(a - cy) + (cy - a)(-b + dx) = 0.\end{aligned}$$

3. (a) A particle is constrained to slide without friction along the curve $y = \alpha x^2$ in the plane (where y represents the vertical direction), under the influence of gravity. Write the Lagrangian $L(\dot{x}, x)$ of the system (using the x coordinate to parametrize the position of the particle) and derive the equation of motion. Remember that the potential energy in a uniform gravitational force field is $U = gy$, where g is the gravitational constant.

Solution. The different quantities associated with the system are easily computed:

$$\begin{aligned}K &= \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) = \tfrac{1}{2}(\dot{x}^2 + 2\alpha x\dot{x}) = \tfrac{1}{2}\dot{x}^2(1 + 4\alpha^2 x^2), \\ U &= gy = \alpha g x^2, \\ L &= K - U = \tfrac{1}{2}\dot{x}^2(1 + 4\alpha^2 x^2) - \alpha g x^2, \\ p &= \frac{\partial L}{\partial \dot{x}} = (1 + 4\alpha^2 x^2)\dot{x}, \\ \frac{\partial L}{\partial x} &= 2x(2\alpha^2 \dot{x}^2 - \alpha g).\end{aligned}$$

The Euler-Lagrange equation becomes

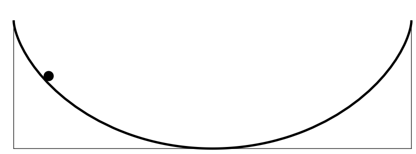
$$\dot{p} = (1 + 4\alpha^2 x^2)\ddot{x} + 8\alpha^2 x\dot{x}^2 = 2x(2\alpha^2\dot{x}^2 - \alpha g),$$

or, after some tidying up,

$$(1 + 4\alpha^2 x^2)\ddot{x} + 4\alpha^2 x\dot{x}^2 + 2\alpha g x = 0.$$

(b) A particle is constrained as in part (a) above to slide along the *inverted cycloid*, which is the curve given in parametric form (in the same coordinate system as above) by the equations

$$\begin{aligned}x(\theta) &= a(\theta - \sin\theta) \\ y(\theta) &= a(1 + \cos\theta)\end{aligned} \qquad (0 \le \theta \le 2\pi),$$



A particle sliding along an inverted cycloid where a is a positive parameter. Let $s = \int_\pi^\theta \sqrt{x'(\phi)^2 + y'(\phi)^2}\, d\phi$ represent the arc length of the cycloid, as measured from the bottom point $\theta = \pi$ of the curve. Write the Lagrangian $L(\dot{s}, s)$, where in this case the generalized coordinate used to track the particle's position is the arc length s. Show that the Euler-Lagrange equation becomes the equation for an harmonic oscillator. Can you think of an interesting physical implication of this result?

Solution. To express the Lagrangian in terms of the coordinate s, we need to find the relationship between the arc length s and the angle parameter θ. We have

$$\begin{aligned} s(\theta) &= \int_\pi^\theta \sqrt{x'(\phi)^2 + y'(\phi)^2}\, d\phi = \int_\pi^\theta a\sqrt{(1-\cos\phi)^2 + \sin^2\phi}\, d\phi \\ &= a\int_\pi^\theta \sqrt{2 - 2\cos\phi}\, d\phi = a\int_\pi^\theta 2\sin(\phi/2)\, d\phi = -4a\cos(\theta/2). \end{aligned}$$

The Lagrangian is the difference between the kinetic and potential energies. The kinetic energy is

$$K = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) = \tfrac{1}{2}\left(\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2\right)\dot{s}^2 = \tfrac{1}{2}\left(\frac{dx^2 + dy^2}{ds^2}\right)\dot{s}^2 = \tfrac{1}{2}\dot{s}^2$$

(a useful point to remember: when a curve is parametrized in terms of arc length, the kinetic energy will *always* be $\frac{1}{2}\dot{s}^2$). The potential energy is given by

$$\begin{aligned} U &= gy = ga(1+\cos\theta) = ga(1 + \cos(2(\theta/2))) \\ &= ga(1 + 2\cos^2(\theta/2) - 1) = 2ga\left(\frac{-s}{4a}\right)^2 = \frac{g}{8a}s^2 \end{aligned}$$

Combining these results, we get that the Lagrangian is

$$L(\dot{q}, q) = \tfrac{1}{2}\dot{s}^2 - \frac{g}{8a}s^2.$$

We recognize that this is the Lagrangian for the harmonic oscillator problem. The generalized momentum p associated with s is $p = \frac{\partial L}{\partial \dot{s}} = \dot{s}$, and the Euler-Lagrange equation is the harmonic oscillator equation

$$\ddot{s} = -\frac{g}{4a}s.$$

One implication of this result is that the period of oscillation of the particle around the midpoint of the curve will not depend on its starting point (since we know that the period of oscillation of an harmonic oscillator is independent of its amplitude). See the Wikipedia article on "tautochrone curve" (`http://en.wikipedia.org/wiki/Tautochrone_curve`) for historical details (and some more mathematical details) about the problem of finding a curve with this property.