Homework due: Friday 4/20 in class

Problems

1. In the computer game "Angry Birds Space," let $\mathbf{x}(t) = (x(t), y(t))$ denote the position of an angry bird at time t flying near a planet centered at $(0,0)$, and denote $r(t) = |\mathbf{x}(t)| = \sqrt{x(t)^2 + y(t)^2}$. Experimental evidence suggests[1] that $\mathbf{x}$ satisfies the equation

$$\ddot{\mathbf{x}} = -F(r)\frac{\mathbf{x}}{r}$$

for some unknown function $F(r)$ (since $\mathbf{x}/r$ is a unit vector, $F(r)$ represents the force of attraction towards the planet). The force function $F(\cdot)$ is a function of a single variable, so we can associate with it a potential function $U(r)$ such that $F(r) = -U'(r)$.

(a) Show that the quantity $M = x\dot{y} - y\dot{x}$ (the "angry bird angular momentum") is conserved along trajectories, i.e., $\frac{dM}{dt} = 0$.

(b) Show that the radial distance $r(t)$ of the angry bird from the center of the planet satisfies the equation

$$\ddot{r} = -U'(r) + \frac{M^2}{r^3},$$

where M is the constant of motion from part (a) above (note that M is a function of the initial conditions $\mathbf{x}(0), \dot{\mathbf{x}}(0)$). An equivalent way of saying this is that r is derived from a conservative system with one degree of freedom $\ddot{r} = -V'(r)$, with effective potential $V(r) = U(r) + \frac{M^2}{2r^2}$.

(c) Let $U(r) = -\frac{k}{r}$, where $k > 0$ (corresponding to the case of ordinary Newtonian gravitation). Find the minimal and maximal radial distances $r_{\min}$ and $r_{\max}$ as a function of k and of the initial conditions $\mathbf{x}(0), \dot{\mathbf{x}}(0)$. This should include a condition for when $r_{\max} = \infty$, i.e., a characterization of when the motion is unbounded.
Hint. Use parts (a) and (b) above as well as the fact that $E = \frac{1}{2}\dot{r}^2 + V(r)$ is a conserved quantity. Try to express $r_{\min}, r_{\max}$ in terms of E and M where possible. You will need to divide into several cases.

(d) Repeat part (c) for a potential of the form $U(r) = \frac{1}{2}kr^2$ $(k > 0)$, corresponding to a two-dimensional harmonic oscillator.

2. The *Lotke-Volterra predator-prey equations* are given by

$$\begin{aligned} \dot{x} &= x(a - cy), \\ \dot{y} &= y(-b + dx), \end{aligned} \qquad (x, y > 0),$$

where a, b, c, d are positive parameters. This planar system of ODEs is used to model the interaction over time of the population sizes (represented by the dynamic variables x, y) of two biological species with one population preying on the other.

(a) Show that the change of variables $p = \log x$, $q = \log y$ transforms the system into a Hamiltonian system $\dot{p} = -\frac{\partial H}{\partial q}, \dot{q} = \frac{\partial H}{\partial p}$, and find the associated Hamiltonian.

[1] see: `http://www.wired.com/wiredscience/2012/03/the-gravitational-force-in-angry-birds-space`

(b) Use the fact that the Hamiltonian you found is autonomous to write down a conserved quantity $G(x,y)$ (of the original system). Note that this gives a family of implicit equations $G(x,y) = C$ which for various values of the constant C describe the shape of the solution curves in the x-y plane.

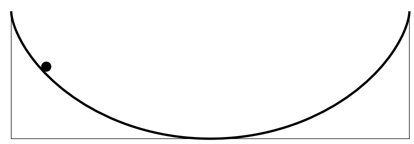
(c) Use a computer and the answer to part (b) above (for some specific choice of parameters, e.g., $a = b = c = d = 1$) to plot the phase portrait of the system. For example, one way to do this is by entering the following text, with `G[x,y]` replaced by the answer to part (b), into the input box on `http://www.wolframalpha.com`:

```
ContourPlot[G[x,y], {x, 0, 5}, {y, 0, 5}]
```

3. (a) A particle is constrained to slide without friction along the curve $y = \alpha x^2$ in the plane (where y represents the vertical direction), under the influence of gravity. Write the Lagrangian $L(\dot{x}, x)$ of the system (using the x coordinate to parametrize the position of the particle) and derive the equation of motion. Remember that the potential energy in a uniform gravitational force field is $U = gy$, where g is the gravitational constant.

(b) A particle is constrained as in part (a) above to slide along the *inverted cycloid*, which is the curve given in parametric form (in the same coordinate system as above) by the equations

$$\begin{aligned} x(\theta) &= a(\theta - \sin\theta) \\ y(\theta) &= a(1 + \cos\theta) \end{aligned} \qquad (0 \le \theta \le 2\pi),$$



A particle sliding along an inverted cycloid

where a is a positive parameter. Let $s = \int_\pi^\theta \sqrt{x'(\phi)^2 + y'(\phi)^2}\, d\phi$ represent the arc length of the cycloid, as measured from the bottom point $\theta = \pi$ of the curve. Write the Lagrangian $L(\dot{s}, s)$, where in this case the generalized coordinate used to track the particle's position is the arc length s. Show that the Euler-Lagrange equation becomes the equation for an harmonic oscillator. Can you think of an interesting physical implication of this result?