1. Find the stationary points of the following functionals:

   (a) $\int_0^1 (q'(x)^2 + 12xq(x))\,dx, \quad q(0) = 0,\ q(1) = 2$

   Solution. The functional is $\int_0^1 L(q', q, x)$ where $L(q', q, x) = q'^2 + 12xq$. We have $\frac{\partial L}{\partial q'} = 2q'$, $\frac{\partial L}{\partial q} = 12x$. The Euler-Lagrange equation is

   $$2q''(x) = 12x.$$

   The general solution is of the form $q(x) = x^3 + C_1 x + C_2$. Imposing the boundary conditions $q(0) = 0, q(1) = 2$ gives that $C_1 = 1, C_2 = 0$, so the stationary point is

   $$q(x) = x^3 + x.$$

   (b) $\int_0^{\pi/2} (q(x)^2 + q'(x)^2 - 2q(x)\sin x)\,dx$ (unknown boundary conditions—find general form of the solution)

   Solution. Here we have $L(q', q, x) = q^2 + q'^2 - 2q\sin x)$, $\frac{\partial L}{\partial q'} = 2q'$, $\frac{\partial L}{\partial q} = 2q - 2\sin x$. The Euler-Lagrange equation is

   $$2q''(x) = 2q(x) - 2\sin x$$

   The general solution has the form

   $$q(x) = Ae^x + Be^{-x} + \tfrac{1}{2}\sin x.$$

   (c) $\int_0^{\pi/2} (y'^2 - y^2 + 2xy)\,dx, \quad y(0) = 0,\ y(\pi/2) = 0$

   Solution. The Euler-Lagrange equation is $y'' = x - y$. Its general solution is $y(x) = A\sin x + B\cos x + x$, and with the boundary conditions $y(0) = 0, y(\pi/2) = 0$ we get that $B = 0, A = -\pi/2$, so the solution is

   $$y = x - \frac{\pi}{2}\sin x.$$

   (d) $\int_2^3 \frac{y'^2}{x^3}\,dx, \quad y(2) = 1,\ y(3) = 16$

   Solution. The general solution to the Euler-Lagrange equation is $y = cx^4 + d$. Imposing the boundary conditions gives $c = 3/13, d = -35/13$, so the solution is

   $$y = \frac{3}{13}x^4 - \frac{35}{13}.$$

2. Show that the functional $\Phi(y) = \int_0^1 (xy + y^2 - 2y^2 y')\,dx$ does not have stationary points subject to the constraints $y(0) = 1, y(1) = 2$.

Solution. Here $L = xy + y^2 - 2y^2y'$. We have

$$\frac{\partial L}{\partial y} = x + 2y - 4yy',$$
$$\frac{\partial L}{\partial y'} = -2y^2,$$
$$0 = \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) - \frac{\partial L}{\partial y} = -4yy' - (x + 2y - 4yy') = x + 2y \quad \text{(Euler-Lagrange)}.$$

The Euler-Lagrange equation in this case is not an ODE but simply the relation $y = -x/2$. This function does not satisfy the boundary conditions $y(0) = 1, y(1) = 2$, so there is no stationary point in this case.

3. The air resistance experienced by a bullet, whose shape is the solid of revolution of a curve $y = q(x)$ moving through the air in the negative x-direction, is

$$\Phi = 4\pi\rho v^2 \int_0^\ell q(x)q'(x)^3\,dx,$$

where ρ is the density of the material, v is the velocity of motion and ℓ is the length of the body of the bullet. Find the optimal shape $q(x)$ that results in the smallest resistance, subject to the conditions $q(0) = 0, q(\ell) = R$.

Hint. Use the identity $\frac{d}{dx}(q(x)q'(x)^3) = q'(x)^4 + 3q(x)q'(x)^2q''(x)$.

Solution. The Euler-Lagrange equation for this functional (ignoring the leading constant $4\pi\rho v^2$) is

$$0 = \frac{d}{dx}\left(\frac{\partial L}{\partial q'}\right) - \frac{\partial L}{\partial q} = \frac{d}{dx}(3qq'^2) - q'^3.$$

Multiplying this equation by q' and rearranging, we get using the identity in the hint,

$$0 = q'\frac{d}{dx}(3qq'^2) - q'^4 = q'(3q'^3 + 6qq'q'') - q'^4 = 2q'^4 + 6qq'^2q''$$
$$= 2(q'^4 + 3qq'^2q'') = 2\frac{d}{dx}\left(qq'^3\right) \implies qq'^3 = \text{const},$$

or, using separation of variables $q^{1/3}dq = C\,dx$, which produces the relation

$$q^{4/3} = Cx + D \implies q(x) = (Cx + D)^{3/4}.$$

Plugging in the boundary conditions $q(0) = 0, q(\ell) = R$ leads to the final answer (illustrated in the figure)

$$q(x) = R\left(\frac{x}{\ell}\right)^{4/3}.$$

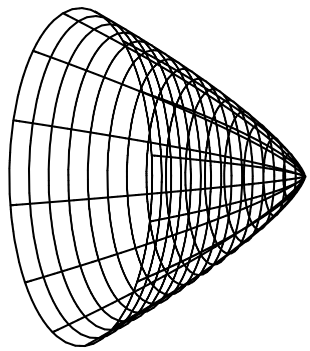

Figure 1: A surface of revolution with minimal air resistance

4. The *Foucault pendulum* is a system with two degrees of freedom $(x(t), y(t))$ satisfying (in the approximation of small oscillations) the equations

$$\ddot{x} = -\omega^2 x + 2\Omega \dot{y},$$
$$\ddot{y} = -\omega^2 y - 2\Omega \dot{x},$$

where ω, Ω are parameters: $\omega = \sqrt{g/\ell}$ is the usual resonant angular frequency associated with a pendulum, and $\Omega = \Omega_0 \sin(\theta)$ where $\Omega_0 = \frac{2\pi}{86400} \frac{\text{rad}}{\text{sec}}$ is the angular frequency of rotation of the earth, and θ is the latitude coordinate in the geographical location where the experiment is performed (e.g., $\theta = 0$ on the equator, $\theta = 90°$ at the north pole, $\theta \approx 38.5°$ in Davis). Typically, $\Omega << \omega$.

(a) Define a new complex-valued coordinate $z = e^{i\Omega t}(x + iy)$ (where $i = \sqrt{-1}$). Show that z satisfies a linear second-order ODE with constant coefficients, and find its general solution. (Note that $z(t)$ keeps track of the pendulum's oscillation in a system of coordinates that rotates at angular velocity Ω relative to the $x - y$ axes.)

Alternative approach: If you are feeling uncomfortable working with ODEs in the complex plane, instead define the new coordinate system (u, v) where u and v are defined by

$$u = \cos(\Omega t)x - \sin(\Omega t)y,$$
$$v = \sin(\Omega t)x + \cos(\Omega t)y.$$

Show that the vector (u, v) satisfies a linear second-order ODE with constant coefficients, and find its general solution.

Solution.

$$\begin{aligned}
z &= e^{i\Omega t}(x+iy), \\
\dot{z} &= i\Omega e^{i\Omega t}(x+iy) + e^{i\Omega t}(\dot{x}+i\dot{y}) = i\Omega z + e^{i\Omega t}(\dot{x}+i\dot{y}), \\
\ddot{z} &= i\Omega(i\Omega z + e^{i\Omega t}(\dot{x}+i\dot{y})) + i\Omega e^{i\Omega t}(\dot{x}+i\dot{y}) + e^{i\Omega t}(\ddot{x}+i\ddot{y}) \\
&= -\Omega^2 z + e^{i\Omega t}\left[i\Omega(\dot{x}+i\dot{y} + i\Omega(\dot{x}+i\dot{y} - \omega^2(x+iy) + 2\Omega(y-ix)\right] \\
&= -(\omega^2+\Omega^2)z.
\end{aligned}$$

Thus, the equation for z (which, remember, is a complex variable, i.e., a two-dimensional vector) is the equation for a two-dimensional harmonic oscillator. If we denote $z = u+iv$ then the real and imaginary parts of z satisfy the equations

$$\begin{aligned}
\ddot{u} &= -\omega_{\text{eff}}^2 u, \\
\ddot{v} &= -\omega_{\text{eff}}^2 v,
\end{aligned}$$

which are uncoupled equations for two one-dimensional harmonic oscillators. These oscillators have a resonant angular frequency $\omega_{\text{eff}} = \sqrt{\omega^2+\Omega^2}$, i.e., the Foucault pendulum oscillates slightly faster at high latitudes than at the equator (or on a planet that is not rotating)—see below. The general solutions are

$$\begin{aligned}
u &= A\cos(\omega_{\text{eff}}t) + B\sin(\omega_{\text{eff}}t), \\
v &= C\cos(\omega_{\text{eff}}t) + D\sin(\omega_{\text{eff}}t),
\end{aligned}$$

where A, B, C, D are arbitrary real numbers, or in complex notation

$$z = z_0 e^{i\omega_{\text{eff}}t} + z_1 e^{-i\omega_{\text{eff}}t},$$

where z_0, z_1 are arbitrary complex numbers.

(b) Explain from the solution to part (a) above how the resonant frequency changes at the north pole. For example, if $\omega = 2\pi$ (corresponding to one oscillation per second—note that the units of ω are radians per second), what is the effective resonant frequency of the oscillations?

Solution. If $\omega = 2\pi$ then the effective angular frequency of oscillations is

$$\omega_{\text{eff}} = \sqrt{\omega^2+\Omega^2} = 2\pi\sqrt{1+\left(\frac{1}{86400}\right)^2} \approx 2\pi\cdot(1+6.69\times10^{-11}).$$

That is, the relative change to the period of oscillation at the north pole (where the effect is strongest) is less than one part in ten billion, and therefore quite negligible.

(c) A Foucault pendulum is set in motion in Davis in the direction of the x-axis (i.e., with initial conditions $x(0) = y(0) = 0, \dot{x}(0) > 0, \dot{y}(0) = 0$.) After 24 hours, what will be the direction of its oscillations relative to the positive x-axis?

Solution. The solution to part (a) shows that the direction of oscillation of the pendulum changes with angular frequency $\Omega = \Omega_0 \sin\theta$. At the north pole the line of oscillation goes through a full rotation every 24 hours. At the latitude of Davis, $\sin(\theta) \approx 0.6225$, so the angle of oscillation relative to the x-axis after 24 hours is

$$\sin(\theta) \cdot 360^\circ \approx 224^\circ.$$