Homework due: Friday 4/27 in class

Problems

1. Find the stationary points of the following functionals:
   (a) $\int_0^1 (q'(x)^2 + 12xq(x))\,dx$, $q(0) = 0$, $q(1) = 2$
   (b) $\int_0^{\pi/2} (q(x)^2 + q'(x)^2 - 2q(x)\sin x)\,dx$ (unknown boundary conditions—find general form of the solution)
   (c) $\int_0^{\pi/2} (y'^2 - y^2 + 2xy)\,dx$, $y(0) = 0$, $y(\pi/2) = 0$
   (d) $\int_2^3 \frac{y'^2}{x^3}\,dx$, $y(2) = 1$, $y(3) = 16$
2. Show that the functional $\Phi(y) = \int_0^1 (xy + y^2 - 2y^2y')\,dx$ does not have stationary points subject to the constraints $y(0) = 1, y(1) = 2$.
3. The air resistance experienced by a bullet, whose shape is the solid of revolution of a curve $y = q(x)$ moving through the air in the negative x-direction, is

$$\Phi = 4\pi\rho v^2 \int_0^\ell q(x)q'(x)^3\,dx,$$

   where ρ is the density of the material, v is the velocity of motion and ℓ is the length of the body of the bullet. Find the optimal shape $q(x)$ that results in the smallest resistance, subject to the conditions $q(0) = 0, q(\ell) = R$.

   Hint. Use the identity $\frac{d}{dx}(q(x)q'(x)^3) = q'(x)^4 + 3q(x)q'(x)^2q''(x)$.

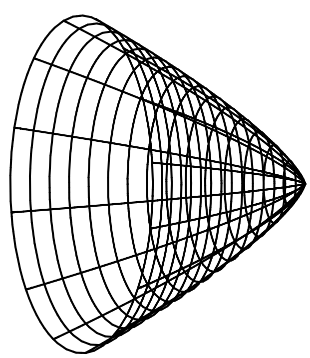

Figure 1: A surface of revolution with minimal air resistance

4. The *Foucault pendulum* is a system with two degrees of freedom $(x(t), y(t))$ satisfying (in the approximation of small oscillations) the equations

$$\ddot{x} = -\omega^2 x + 2\Omega\dot{y},$$
$$\ddot{y} = -\omega^2 y - 2\Omega\dot{x},$$

where ω, Ω are parameters: $\omega = \sqrt{g/\ell}$ is the usual resonant angular frequency associated with a pendulum, and $\Omega = \Omega_0 \sin(\theta)$ where $\Omega_0 = \frac{2\pi}{86400} \frac{\text{rad}}{\text{sec}}$ is the angular frequency of rotation of the earth, and θ is the latitude coordinate in the geographical location where the experiment is performed (e.g., $\theta = 0$ on the equator, $\theta = 90°$ at the north pole, $\theta \approx 38.5°$ in Davis). Typically, $\Omega << \omega$.

(a) Define a new complex-valued coordinate $z = e^{i\Omega t}(x + iy)$ (where $i = \sqrt{-1}$). Show that z satisfies a linear second-order ODE with constant coefficients, and find its general solution. (Note that $z(t)$ keeps track of the pendulum's oscillation in a system of coordinates that rotates at angular velocity Ω relative to the $x - y$ axes.)

Alternative approach: If you are feeling uncomfortable working with ODEs in the complex plane, instead define the new coordinate system (u, v) where u and v are defined by

$$u = \cos(\Omega t)x - \sin(\Omega t)y,$$
$$v = \sin(\Omega t)x + \cos(\Omega t)y.$$

Show that the vector (u, v) satisfies a linear second-order system of ODEs with constant coefficients, and find its general solution.

(b) (Optional) Explain from the solution to part (a) above how the resonant frequency changes at the north pole. For example, if $\omega = 2\pi$ (corresponding to one oscillation per second—note that the units of ω are radians per second), what is the effective resonant frequency of the oscillations?

(c) (Optional) A Foucault pendulum is set in motion in Davis in the direction of the x-axis (i.e., with initial conditions $x(0) = y(0) = 0, \dot{x}(0) > 0, \dot{y}(0) = 0$.) After 24 hours, what will be the direction of its oscillations relative to the positive x-axis?