1. A rope of length ℓ and uniform linear density ρ hangs between two points $(-A/2, 0)$, $(A/2, 0)$, where we assume that $\ell \geq A$. Its shape $y = q(x)$ is determined by the requirement that the potential energy

$$E = -\rho g \int_{-A/2}^{A/2} q(x)\, ds = -\rho g \int_{-A/2}^{A/2} q(x)\sqrt{1 + q'(x)^2}\, dx$$

is minimized (here, $ds = \sqrt{dx^2 + dy^2}$ is the arc length element), subject to the constraints

$$\ell = \int_{-A/2}^{A/2} \sqrt{1 + q'(x)^2}\, dx, \quad q(-A/2) = q(A/2) = 0.$$

Show that the equation for the shape (known as a *catenary*) is

$$q(x) = c + r\cosh(x/r)$$

for suitable constants c and r which can be expressed as functions of ℓ and A.

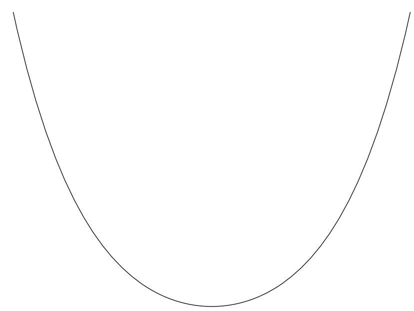


Figure 1: The shape of a hanging rope

Solution. Following the instructions in the hint, the modified Lagrangian is

$$L_\lambda(q', q) = q\sqrt{1 + q'^2} + \lambda\sqrt{1 + q'^2} = (q + \lambda)\sqrt{1 + q'^2}.$$

The minimizing curve q satisfies the Euler-Lagrange equation, and in this case (since the Lagrangian is autonomous), it also satisfies the energy conservation equation

$$\frac{dH}{dt} = \frac{d}{dt}\left(pq' - L_\lambda\right) = 0,$$

where $p = \frac{\partial L_\lambda}{\partial q'} = \frac{(q+\lambda)q'}{\sqrt{1+q'^2}}$. So we get the first-order ODE

$$\begin{aligned} 0 &= \frac{d}{dx}\left(\frac{(q+\lambda)q'^2}{\sqrt{1+q'^2}} - (q+\lambda)\sqrt{1+q'^2}\right) \\ &= \frac{d}{dx}\left((q+\lambda)\left(\frac{q'^2 - (1+q'^2)}{\sqrt{1+q'^2}}\right)\right) = -\frac{d}{dx}\left(\frac{q+\lambda}{\sqrt{1+q'^2}}\right), \end{aligned}$$

or

$$\frac{q+\lambda}{\sqrt{1+q'^2}} = k \quad \Longrightarrow \quad q+\lambda = k\sqrt{1+q'^2},$$

where λ is the Lagrange multiplier (an arbitrary number that can be chosen to fit the boundary conditions) and k is an arbitrary integration constant. It is now easy to check that the formula for the catenary, $q(x) = r\cosh(x/r) + c$, satisfies this equation, since we have

$$\sqrt{1+q'^2} = \sqrt{1+\sinh(x/r)^2} = \cosh(x/r) = \frac{q-c}{r}.$$

Finally, having found the general form of the solution (the family of catenary curves), by imposing the boundary conditions $q(-A/2) = q(A/2) = 0$ and the arc length constraint $\ell = \int ds$ we can find the values for the parameters r and c in terms of A and ℓ. The value of r is the solution to the equation

$$\ell = \int_{-A/2}^{A/2} \sqrt{1+\sinh^2(x/r)}\,dx = \int_{-A/2}^{A/2} \cosh(x/r)\,dx = 2r\sinh(A/2r),$$

and the value of c is

$$\begin{aligned} c &= q(A/2) - r\cosh(A/2r) = -r\cosh(A/2r) = -r\sqrt{1+\sinh^2(A/2r)} \\ &= -r\sqrt{1+\frac{\ell^2}{4r^2}} = -\sqrt{r^2+(\ell/2)^2}. \end{aligned}$$

2. The exponential $e^A = \exp(A)$ of a square matrix $A = (a_{i,j})_{i,j=1}^d$ is defined by

$$\exp(A) = \sum_{n=0}^{\infty} \frac{1}{n!} A^n.$$

(a) Show that if P and A are square matrices and P is invertible then

$$\exp(PAP^{-1}) = Pe^A P^{-1}.$$

Solution. Note that for any $n \geq 1$,

$$(PAP^{-1})^n = PAP^{-1} \cdot PAP^{-1} \cdot \ldots \cdot PAP^{-1} = P(AA\ldots A)P^{-1} = PA^nP^{-1}.$$

It follows that

$$\begin{aligned} \exp(PAP^{-1}) &= I + (PAP^{-1}) + \frac{1}{2!}(PAP^{-1})^2 + \frac{1}{3!}(PAP^{-1})^3 + \ldots \\ &= PP^{-1} + PAP^{-1} + \frac{1}{2!}PA^2P^{-1} + \frac{1}{3!}PA^3P^{-1} + \ldots \\ &= P\left(I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \ldots\right)P^{-1} = Pe^AP^{-1}. \end{aligned}$$

(b) Compute e^{tA} for the following matrices A:

i. $A = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$

Solution. $e^{tA} = \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{pmatrix}$

ii. $A = \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix}$

Solution. Diagonalizing A, we find that it has eigenvalues $\lambda_1 = 2$, $\lambda_2 = 3$, with corresponding eigenvectors $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $v_3 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. Equivalently, this means that A can be expressed as $A = PDP^{-1}$, where

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}.$$

Therefore by part (a) above, we have

$$\begin{aligned} \exp(tA) = Pe^{tD}P^{-1} &= \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 3e^{2t} - 2e^{3t} & -e^{2t} + e^{3t} \\ 6e^{2t} - 6e^{3t} & -2e^{3t} + 3e^{3t} \end{pmatrix}. \end{aligned}$$

iii. $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Solution. Observe that $A^2 = -I$. We can therefore write the power series expansion as

$$\begin{aligned} e^t A &= I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \frac{t^4}{4!}A^4 + \dots \\ &= I + tA + \frac{t^2}{2!}(-I) + \frac{t^3}{3!}(-A) + \frac{t^4}{4!}I + \frac{t^5}{5!}A + \dots \\ &= I\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \dots\right) + A\left(1 - \frac{t^3}{3!} + \frac{t^5}{5!} - \dots\right) \\ &= \cos t \cdot I + \sin t \cdot A = \begin{pmatrix} \cos t & 0 \\ 0 & \cos t \end{pmatrix} + \begin{pmatrix} 0 & \sin t \\ -\sin t & 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \end{aligned}$$

(c) Prove that if A, B are commuting square matrices (i.e., matrices which satisfy $AB = BA$) then $\exp(A + B) = \exp(A)\exp(B)$.

Solution.

$$\begin{aligned}
\exp(A+B) &= \sum_{n=0}^{\infty} \frac{1}{n!}(A+B)^n \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} A^k B^{n-k} \qquad = [\text{now denoting } j = n-k] \\
&= \sum_{n=0}^{\infty} \sum_{\substack{j,k\geq 0 \\ j+k=n}} \frac{1}{k!\,j!} A^k B^j = \sum_{k=0}^{\infty} \frac{1}{k!} A^k \cdot \sum_{j=0}^{\infty} \frac{1}{j!} B^j = e^A e^B.
\end{aligned}$$

3. Prove the formula

$$\det\left(\exp(A)\right) = e^{\mathrm{tr}(A)},$$

where A is a square matrix and $\mathrm{tr}(A)$ denotes the trace of A. You may assume that the matrix is of order 2×2 and that $\exp(A)$ has the following equivalent definition:

$$\exp(A) = \lim_{n\to\infty} \left(I + \frac{A}{n}\right)^n$$

(for bonus points, show that this definition is equivalent to the definition using power series and/or prove the claim for a general $k \times k$ matrix).

Solution. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then

$$\begin{aligned}
\det\exp(A) &= \lim_{n\to\infty} \det\left[\left(I + \frac{A}{n}\right)^n\right] = \lim_{n\to\infty} \left(\det\left(I + \frac{A}{n}\right)\right)^n \\
&= \lim_{n\to\infty} \left((1+a/n)(1+d/n) - bc/n^2\right)^n \\
&= \lim_{n\to\infty} \left(1 + \frac{a+d}{n} + \frac{ad-bc}{n^2}\right)^n = e^{a+d} = e^{\mathrm{tr}(A)}.
\end{aligned}$$

Here, we use the well-known limit from calculus: if $(a_n)_{n=1}^{\infty}$ is a sequence of real numbers such that $a_n \to x$, then

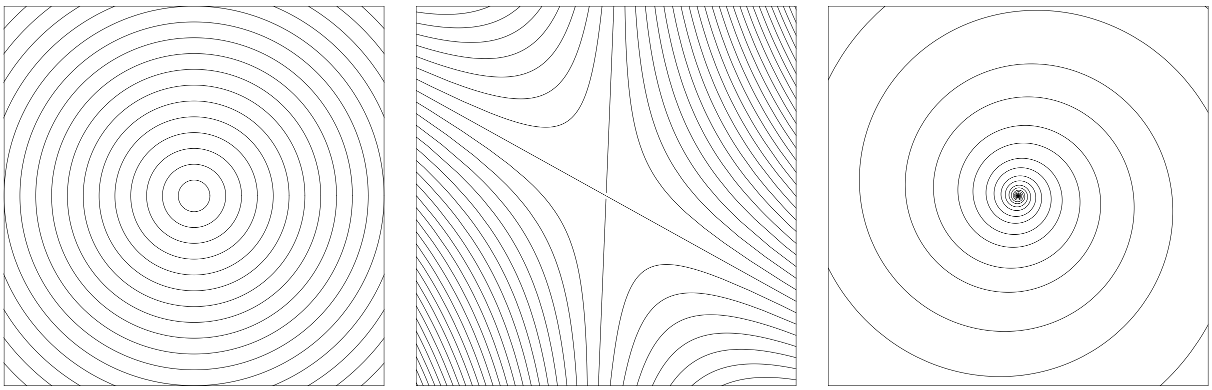
$$\lim_{n\to\infty} \left(1 + \frac{a_n}{n}\right)^n \to e^x.$$

To extend this to matrices of arbitrary order $k \times k$, one needs to explain why the determinant of $I + A/n$ can be written as

$$\det\left(I + \frac{A}{n}\right) = 1 + \frac{trA}{n} + \left[\text{lower order terms} \propto \frac{1}{n^2}\right],$$

after which the same calculation can be repeated as above. To explain this, expand the determinant in the usual way as a sum over $k!$ permutations of $1, 2, \ldots, k$, and notice that the only terms which are of order $1/n$ come from choosing $k-1$ of the diagonal "1" terms from the identity matrix I and the remaining term from the remaining diagonal term of A/n. The sum of these terms is exactly $\mathrm{tr}(A)/n$, and all other terms involve a multiple of at least two terms of A/n and are therefore of order at most $1/n^2$.

4. Which of the phase portraits (a)–(c) shown below can represent the behavior of a Hamiltonian system near a rest point? Explain.



(a) (b) (c)

Solution. The phase portraits (a) and (b) show a center and a saddle point, respectively, which can occur in a Hamiltonian system. The phase portrait (c) shows an asymptotically stable rest point, which we have seen cannot occur (this can be shown either from Liouville's theorem or by analyzing the eigenvalues of the Hessian matrix at a rest point).