Homework due: Friday 5/4 in class

Problems

1. A rope of length ℓ and uniform linear density ρ hangs between two points $(-A/2, 0)$, $(A/2, 0)$, where we assume that $\ell \geq A$. Its shape $y = q(x)$ is determined by the requirement that the potential energy

$$E = -\rho g \int_{-A/2}^{A/2} q(x)\, ds = -\rho g \int_{-A/2}^{A/2} q(x) \sqrt{1 + q'(x)^2}\, dx$$

is minimized (here, $ds = \sqrt{dx^2 + dy^2}$ is the arc length element), subject to the constraints

$$\ell = \int_{-A/2}^{A/2} \sqrt{1 + q'(x)^2}\, dx, \quad q(-A/2) = q(A/2) = 0.$$

Show that the equation for the shape (known as a *catenary*) is

$$q(x) = c + r \cosh(x/r)$$

for suitable constants c and r which can be expressed as functions of ℓ and A.

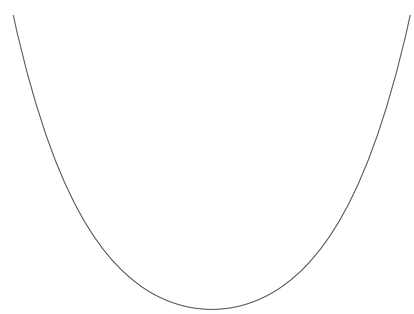


Figure 1: The shape of a hanging rope

Hint. This is a constrained optimization problem. To solve it, form the modified Lagrangian $L_\lambda(q', q) = L + \lambda G$ where L is the original Lagrangian and G is the constraint function. Note that the Lagrangian is "autonomous" (independent of the variable x, which here represents space rather than time) so instead of the Euler-Lagrange equation, one can solve the simpler "conservation of the Hamiltonian" equation

$$\frac{d}{dx}\left(\frac{\partial L_\lambda}{\partial q'} q' - L_\lambda\right) = 0.$$

2. The exponential $e^A = \exp(A)$ of a square matrix $A = (a_{i,j})_{i,j=1}^d$ is defined by

$$\exp(A) = \sum_{n=0}^{\infty} \frac{1}{n!} A^n.$$

(a) Show that if P and A are square matrices and P is invertible then

$$\exp(PAP^{-1}) = P\exp(A)P^{-1}.$$

(b) Compute e^{tA} for the following matrices A:

i. $A = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$

ii. $A = \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix}$

Hint. Diagonalize.

iii. $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Hint. Don't bother diagonalizing—use the power series expansion, and the fact that $A^2 = -I$.

(c) Prove that if A, B are commuting square matrices (i.e., matrices which satisfy $AB = BA$) then $\exp(A + B) = \exp(A)\exp(B)$.

Hint. Expand $\exp(A + B)$ as a power series in $A + B$. For each power $(A + B)^n$, use the binomial identity

$$(A + B)^n = \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k},$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ are the binomial coefficients (the identity is valid because A and B commute so one can treat them like ordinary algebraic variables), then by rearranging the order of summation of the double sum show that it is equal to $\exp(A)\exp(B)$.

3. Prove the formula

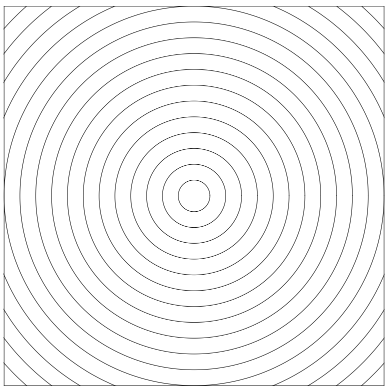
$$\det\left(\exp(A)\right) = e^{\mathrm{tr}(A)},$$

where A is a square matrix and $\mathrm{tr}(A)$ denotes the trace of A. You may assume that the matrix is of order 2×2 and that $\exp(A)$ has the following equivalent definition:

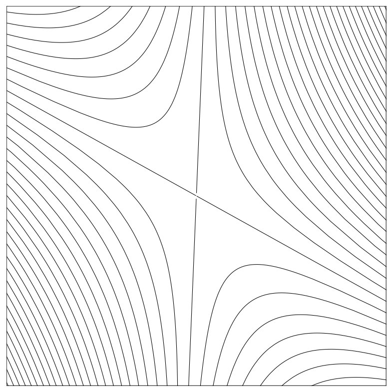
$$\exp(A) = \lim_{n\to\infty} \left(I + \frac{A}{n}\right)^n$$

(for bonus points, show that this definition is equivalent to the definition using power series and/or prove the claim for a general $k \times k$ matrix).

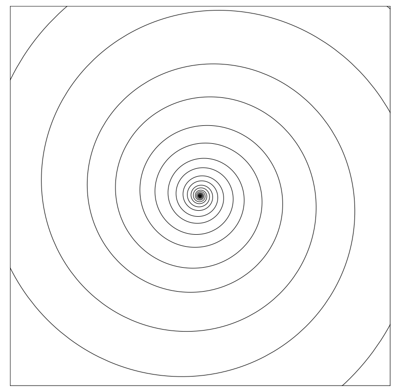
4. Which of the phase portraits (a)–(c) shown below can represent the behavior of a Hamiltonian system near a rest point? Explain.



(a)



(b)



(c)