1. Find the solution x_n to the recurrence

$$x_{n+2} = x_{n+1} + 2x_n$$

satisfying the initial conditions $x_1 = 5$, $x_2 = 1$.

Solution. As in the example of the Fibonacci numbers discussed in class, we can express the problem in terms of a planar dynamical system transforming a column vector v via the matrix multiplication equation

$$v_{n+1} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} v_n = M v_n.$$

If we denote $v_n = \begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix}$, then we see that

$$\begin{pmatrix} x_{n+2} \\ x_{n+1} \end{pmatrix} = v_{n+1} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} x_{n+1} + 2x_n \\ x_{n+1} \end{pmatrix},$$

so the first-order planar dynamics reproduces the second-order recurrence we started with. With this matrix representation we immediately see that

$$v_n = M v_{n-1} = M^2 v_{n-2} = \ldots = M^{n-1} v_1,$$

where $v_1 = (x_2, x_1)^\top = (5, 1)^\top$. If M can be diagonalized, that is, we have $M = PDP^{-1}$ where P is an invertible matrix and D is diagonal, then we get that

$$v_n = M^{n-1} v_1 = P D^{n-1} P^{-1} v_1.$$

Since D^{n-1} is also a diagonal matrix with the numbers λ_1^{n-1} and λ_2^{n-1} in the diagonal (where λ_1, λ_2 are the two eigenvalues of M), we can immediately see without any further computation that each of the entries of v_n will be a linear combination of the two exponential terms λ_1^n and λ_2^n. In other words, we have

$$x_n = a\lambda_1^n + b\lambda_2^n$$

for some constants a, b. It remains to compute λ_1, λ_2 and find the coefficients a, b. The eigenvalues of M are easily found to be 2 and -1, giving

$$x_n = a \cdot 2^n + b(-1)^n.$$

Letting $n = 1$ and $n = 2$ gives two equations for a, b from the initial conditions:

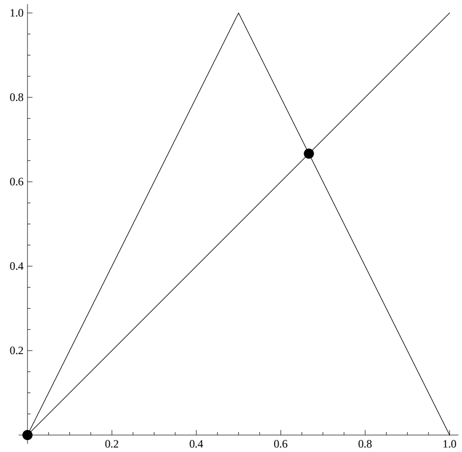
$$5 = x_1 = 2a - b,$$
$$1 = x_2 = 4a + b.$$

The solution of this pair of equations is $a = 1, b = -3$, so the final answer is

$$x_n = 2^n - 3(-1)^n.$$

2. Find all fixed points and all 2-cycles of the tent map $\Lambda_r(x)$ in the case $r = 2$.

Solution. Using the graphical method for intuition, to find the fixed points we plot the function $y = \Lambda_2(x)$ together with $y = x$:

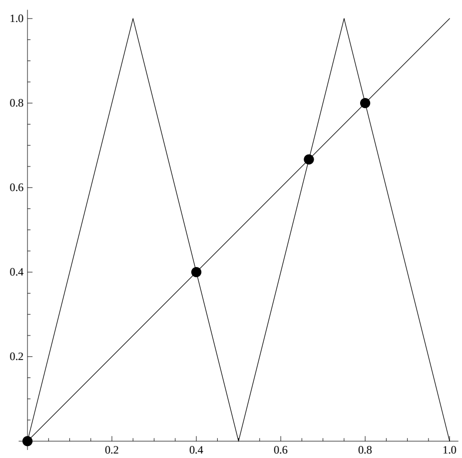


We see that there are two fixed points, one at $x = 0$ and another at the point which solves the equation $x = 2 - 2x$, namely $x = 2/3$.

Next, for the 2-cycles, we do the same with the iterated map $\Lambda_2 \circ \Lambda_2$. The formula for $\Lambda_2 \circ \Lambda_2$ is divided into 4 expressions, according to which of the 4 intervals $[0, 1/4]$, $[1/4, 1/2]$, $[1/2, 3/4]$, $[3/4, 1]$ the point x falls in. A short computation gives:

$$\Lambda_2 \circ \Lambda_2(x) = \begin{cases} 4x & \text{if } 0 \le x \le 1/4, \\ 2 - 4x & \text{if } 1/4 \le x \le 1/2, \\ 4x - 2 & \text{if } 1/2 \le x \le 3/4, \\ 4 - 4x & \text{if } 3/4 \le x \le 1. \end{cases}$$

Plotting this function together with $y = x$ gives the following picture:



There are 4 intersection points: two are the fixed points $x = 0$ and $x = 2/3$ which we already found, which are not parts of a 2-cycle, and the other two are at $x_1 = 2/5$ (the solution of the equation $x = 2 - 4x$) and $x_2 = 4/5$ (solution of $x = 4 - 4x$). These two points form a 2-cycle, since $\Lambda_2(x_1) = x_2, \Lambda_2(x_2) = x_1$. There are no other 2-cycles.

3. For which values of $0 < \alpha < 1$ does the circle rotation map R_α have a 2-cycle? For which values does it have a 3-cycle?

   Solution. Let $0 < \alpha < 1$. If x is a member of a 2-cycle of R_α, then

   $$x = R_\alpha(R_\alpha(x)) = x + 2\alpha \bmod 1.$$

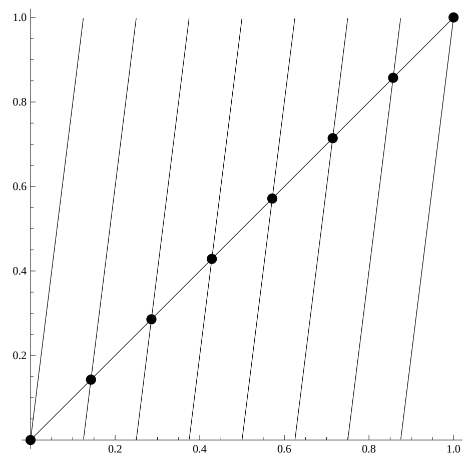
   Because of the geometric interpretation of R_α as a circle rotation map, where x represents a point on the unit circle (corresponding to the angular coordinate $\theta = 2\pi x$, and α represents a rotation by an angle $2\pi\alpha$, the only way we could come back to $2\pi x$ after two rotations by an angle $2\pi\alpha$ is if $\alpha = 1/2$. Note that for this choice of α, *every* $x \in [0,1)$ satisfies $R_\alpha(R_\alpha(x)) = x$, i.e., every x is a member of a 2-cycle $(x, R_\alpha(x))$.

   Similarly, it is easy to see that the only ways to get a 3-cycle are to take $\alpha = 1/3$ (rotation by an angle $2\pi/3$) or $\alpha = 2/3$ (rotation by an angle $4\pi/3$). In these cases, every $x \in [0,1)$ belongs to a 3-cycle $(x, R_\alpha(x), R_\alpha(R_\alpha(x)))$.

4. (a) Sketch the graph of the third iteration $D^3 = D \circ D \circ D$ of the doubling map $D(x) = 2x \bmod 1$ on the interval $(0,1)$. Use this to find all its 3-cycles. **Solution.** The third iteration of D is

$$D^3(x) = 8x \bmod 1 = \begin{cases} 8x & \text{if } 0 \le x \le 1/8, \\ 8x-1 & \text{if } 1/8 \le x \le 1/4, \\ 8x-2 & \text{if } 1/4 \le x \le 3/8, \\ & \vdots \\ 8x-7 & \text{if } 7/8 \le x \le 1. \end{cases}$$

Plotting it together with $y = x$ gives the following picture



which shows that we get expect 8 intersection points. They are the solutions of the equations $x = 8x$, $x = 8x-1$, $x = 8x-2$, $\ldots$, $x = 8x-7$, i.e. they take the form $x_k = k/7$, $k = 0,1,\ldots,7$. For $k = 0$ and $k = 1$ these are just the points $x = 0, 1$ which are fixed points of D (and hence not considered as members of 3-cycles). The remaining points split into two 3-cycles: $(1/7, 2/7, 4/7)$ and $(3/7, 6/7, 5/7)$.

(b) Generalize this to find all the k-cycles of D for arbitrary values of k.

Solution. Since $D^k(x) = 2^k x \bmod 1$, we are led to a similar set of equations

$$x = 2^k x - j, \qquad j = 0,1,2,\ldots,2^k,$$

whose solutions are

$$x_j = \frac{j}{2^k - 1}, \qquad j = 0,1,2,\ldots,2^k.$$

In order for a point x_j belongs to a genuine k-cycle, it can't be part of a d-cycle for a divisor d of k (for example, $x = 1/7 = 1/(2^3-1) = 9/(2^6-1)$ belongs to a 3-cycle, therefore it is not a member of a 6-cycle even though it comes up in the computation for $k = 6$). It is possible to show that $x_j = j/(2^k-1)$ is a member of a k-cycle precisely if j is not a multiple of $(2^k-1)/(2^d-1)$ for any divisor $d > 1$ of k.

5. (a) A discrete-time dynamical system on $\mathbb{R}_+ = [0, \infty)$ is defined using the evolution equation

$$x_{n+1} = \sqrt{2 + x_n}.$$

Find the unique fixed point x_* of the map, and show that for any initial condition x_0, we have the limit $x_n \to x_*$ as $n \to \infty$.

Solution. The fixed point satisfies $x_* = \sqrt{2 + x_*}$, or $x_*^2 - x_* - 2 = 0$. This equation has two solutions, -1 and 2, but the dynamical system only acts on nonnegative numbers, so $x_* = 2$ is the unique fixed point in that range.

Assume $x_0 < x_*$. Let us prove by induction that $x_n < x_{n+1}$ for all n, i.e., the sequence is increasing. For $n = 0$ this is true, since $x_1 = \sqrt{2 + x_0} > x_0$ for x_0 in the range $[0, 2)$. Next, if we showed that $x_n > x_{n-1}$, then applying the function $f(x) = \sqrt{2 + x}$ (which is an increasing function) to both sides of this inequality gives

$$x_{n+1} = f(x_n) > f(x_{n-1}) = x_n,$$

which completes the inductive step.

Next, we claim that if $x_0 < 2$ then $x_n < 2$ for all n. The proof is again by induction. The induction base $n = 0$ is just the original assumption, and for the inductive step, if we showed that $x_{n-1} < 2$ then again applying $f(x)$ to both sides gives

$$x_n = f(x_{n-1}) < f(2) = 2,$$

as claimed.

The sequence $(x_n)_{n=0}^{\infty}$ is increasing and bounded from above. We conclude from a standard theorem in real analysis that it is convergent. Denote its limit by x_∞. We have

$$f(x_\infty) = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n = x_\infty.$$

That is, x_∞ is a fixed point of $f(x)$, and since $x_\infty \geq 0$, the only possibility is $x_\infty = 2$. The second case in which $x_0 > 2$ is done analogously—in this case the sequence x_n will decrease monotonically and will be bounded from below by $x_* = 2$, and hence will converge to a limit, which must be again be a fixed point of $f(x)$.

(b) Fill in the blank:

$$\sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \ldots}}}} = \quad ?$$

and, if you can, explain why the question makes sense (i.e., does any weird expression that mathematicians can dream up with "..." have a well-defined value? If not, why does this one?)

Solution. The natural interpretation for expressions involving "..." is in terms of a limiting process involving iteration, exactly as discussed above in terms of iterations of the map $f(x) = \sqrt{2 + x}$. The above proof of convergence shows that the only sensible value that can be attributed to the expression on the left is the fixed point $x_* = 2$. Thus we have:

$$\sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \ldots}}}} = 2.$$

There are many processes which don't converge to a limit, so one can make up "formulas" involving "..." that can't be assigned a well-defined value, e.g., $1 - 2 + 3 - 4 + 5 - 6 + \ldots$