Homework due: Friday 5/18 in class

Problems

1. Find the solution x_n to the recurrence

$$x_{n+2} = x_{n+1} + 2x_n$$

satisfying the initial conditions $x_1 = 5$, $x_2 = 1$.

Hint. The solution takes the form $x_n = a\lambda_1^n + b\lambda_2^n$ where λ_1, λ_2 are eigenvalues of a certain 2×2 matrix.

2. Find all fixed points and all 2-cycles of the tent map $\Lambda_r(x)$ in the case $r = 2$.

3. For which values of $0 < \alpha < 1$ does the circle rotation map R_α have a 2-cycle? For which values does it have a 3-cycle?

4. (a) Sketch the graph of the third iteration $D^3 = D \circ D \circ D$ of the doubling map $D(x) = 2x \bmod 1$ on the interval $(0, 1)$. Use this to find all its 3-cycles.

   (b) (Optional) Generalize this to find all the k-cycles of D for arbitrary values of k.

5. (a) A discrete-time dynamical system on $\mathbb{R}_+ = [0, \infty)$ is defined using the evolution equation

$$x_{n+1} = \sqrt{2 + x_n}.$$

   Find the unique fixed point x_* of the map, and show that for any initial condition x_0, we have the limit $x_n \to x_*$ as $n \to \infty$.

   Hint. Divide into two cases: $x_0 < x_*$ and $x_0 > x_*$ (the remaining case $x_0 = x_*$ is trivial). For the case $x_0 < x_*$, show by induction that for all $n \geq 0$, $x_{n+1} > x_n$, i.e., x_n is an increasing sequence, and that $x_n < x_*$, i.e., the sequence is bounded from above. A result in analysis says that a sequence of real numbers that is increasing and bounded from above must converge to a limit L. Show that L must be equal to x_*. For the second case $x_0 > x_*$ argue analogously.

   (b) Fill in the blank:

$$\sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \ldots}}}} = \quad ?$$

   and, if you can, explain why the question makes sense (i.e., does any weird expression that mathematicians can dream up with "..." have a well-defined value? If not, why does this one?)

6. (Optional) Let T be the $3x + 1$ map on the natural numbers, defined by

$$T(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even,} \\ 3x + 1 & \text{if } x \text{ is odd.} \end{cases}$$

Let $f(x)$ denote the number of iterations of the map needed to get to 1 starting from x:

$$f(x) = \min\{k \geq 0 \,:\, T^k(x) = 1\}.$$

(Note that the Collatz conjecture is equivalent to the statement that $f(x) < \infty$ for all x.) Write a computer program to compute $f(x)$. Plot the values of $f(x)$ for $1 \leq x \leq 1000$ and estimate how fast we may expect $f(x)$ to grow as a function of x.

Additional review problems to help you study for the quiz

(not part of the homework — solutions will be published on Tuesday)

7. Use matrix exponentials, or any other method, to find the solution $(x(t), y(t))$ of the ODE system

$$\begin{aligned} \dot{x} &= 2x + \ \ y, \\ \dot{y} &= \ \ x + 2y \end{aligned}$$

satisfying the initial conditions $x(0) = 0, y(0) = 1$.

8. Compute e^{tA} (don't ignore the t factor!) for the following matrices:

   (a) $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

   (b) $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

   (c) $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

   Hint. This A is not diagonalizable. Instead, write A as $A = I + M$ where M is a matrix which can be seen to satisfy $M^2 = 0$, then work directly with the power series definition $e^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n = \sum_{n=0}^{\infty} \frac{t^n}{n!} (I + M)^n$.

   (d) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

   Hint. Before you compute, think how e^A acts on a matrix in block-diagonal form, i.e., a matrix of the form $\begin{pmatrix} M_1 & \\ & M_2 \end{pmatrix}$, where M_1, M_2 are both square matrices.

9. The planar system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\sin x - \alpha y + I, \end{aligned}$$

where $\alpha, I > 0$ are parameters, describes the dynamics for a simple pendulum with a damping factor α and a constant driving torque I. (The exact same equation models the *Josephson junction*, an important quantum-mechanical system comprised of two weakly coupled semiconductors.)

   (a) Is the above system Hamiltonian?

   (b) Let $\varphi_t : \mathbb{R}^2 \to \mathbb{R}^2$ denote the phase flow map of the system. Note that φ_t is a vector made up of two component functions, i.e., we can denote $\varphi_t(x, y) = (u_t(x, y), v_t(x, y))$. Compute the Jacobian

$$J_t(x, y) = \det \begin{pmatrix} \frac{\partial u_t}{\partial x} & \frac{\partial v_t}{\partial x} \\ \frac{\partial u_t}{\partial y} & \frac{\partial v_t}{\partial y} \end{pmatrix}.$$

   From this computation, try to think what you can conclude about what the phase flow does to the phase space (e.g., is area conserved? Will a small area become large as it carried along by the flow, or vice versa?)