7. Use matrix exponentials, or any other method, to find the solution $(x(t), y(t))$ of the ODE system

$$\begin{aligned} \dot{x} &= 2x + \ \ y, \\ \dot{y} &= \ \ x + 2y \end{aligned}$$

satisfying the initial conditions $x(0) = 0, y(0) = 1$.

Solution. The system has the form $(\dot{x}, \dot{y})^\top = A(x, y)^\top$, where A is the 2×2 matrix $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, so the solution for the given initial conditions will be

$$\begin{pmatrix} x(t) \\ y(y) \end{pmatrix} = e^{tA} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

To compute the matrix exponential e^{tA}, we diagonalize A. Its eigenvalues are found to be $\lambda_1 = 1, \lambda_2 = 3$, with associated eigenvectors $v_1 = (1, -1)^\top, v_2 = (1, 1)^\top$. This means that we can write $A = PDP^{-1}$ where

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \qquad P = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \qquad P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

It follows that

$$e^{tA} = Pe^{tD}P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} e^t & 0 \\ 0 & e^{3t} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^t + e^{3t} & -e^t + e^{3t} \\ -e^t + e^{3t} & e^t + e^{3t} \end{pmatrix}.$$

The solution $(x(t), y(t))^\top$ is therefore

$$\begin{pmatrix} x(t) \\ y(y) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^t + e^{3t} & -e^t + e^{3t} \\ -e^t + e^{3t} & e^t + e^{3t} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(-e^t + e^{3t}) \\ \frac{1}{2}(e^t + e^{3t}) \end{pmatrix}.$$

8. Compute e^{tA} (don't ignore the t factor!) for the following matrices:

(a) $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Solution. $e^{tA} = I$.

(b) $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

Solution. $e^{tA} = \begin{pmatrix} \frac{e^{2t}+1}{2} & \frac{e^{2t}-1}{2} \\ \frac{e^{2t}-1}{2} & \frac{e^{2t}+1}{2} \end{pmatrix}$.

(c) $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Solution. $A = I + M$ where $M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ satisfies $M^2 = 0$.

$$\begin{aligned} e^{tA} &= \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n = \sum_{n=0}^{\infty} \frac{t^n}{n!} (I+M)^n \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} (I + nM + [\text{higher powers of } M]) \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} (I + nM) = \begin{pmatrix} \sum_{n=0}^{\infty} \frac{t^n}{n!} & \sum_{n=1}^{\infty} \frac{t^n}{(n-1)!} \\ 0 & \sum_{n=0}^{\infty} \frac{t^n}{n!} \end{pmatrix} \\ &= \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix} \end{aligned}$$

(d) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

Solution. The matrix is in block diagonal form, and the matrix exponential acts on each diagonal block separately, so we have

$$e^{tA} = \begin{pmatrix} e^t & \\ & \exp\left(t \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\right) \end{pmatrix} = \begin{pmatrix} e^t & 0 & 0 \\ 0 & \frac{e^t+e^{3t}}{2} & \frac{-e^t+e^{3t}}{2} \\ 0 & \frac{-e^t+e^{3t}}{2} & \frac{e^t+e^{3t}}{2} \end{pmatrix}.$$

(The computation for the 2×2 block appears in the solution to problem 7 above.)

9. The planar system

$$\dot{x} = y,$$
$$\dot{y} = -\sin x - \alpha y + I,$$

where $\alpha, I > 0$ are parameters, describes the dynamics for a simple pendulum with a damping factor α and a constant driving torque I. (The exact same equation models the *Josephson junction*, an important quantum-mechanical system comprised of two weakly coupled semiconductors.)

(a) Is the above system Hamiltonian?

Solution. Denote $F(x, y) = y, G(x, y) = -\sin x - \alpha y + I$, so the system has the general form $\dot{x} = F(x, y), \dot{y} = G(x, y)$. We know such a system is Hamiltonian if and only if $\mathrm{div}(F, G) = \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = 0$. In this case $\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = 0 - \alpha \neq 0$, so the system is not Hamiltonian.

(b) Let $\varphi_t : \mathbb{R}^2 \to \mathbb{R}^2$ denote the phase flow map of the system. Note that φ_t is a vector made up of two component functions, i.e., we can denote $\varphi_t(x, y) = (u_t(x, y), v_t(x, y))$. Compute the Jacobian

$$J_t(x, y) = \det \begin{pmatrix} \frac{\partial u_t}{\partial x} & \frac{\partial v_t}{\partial x} \\ \frac{\partial u_t}{\partial y} & \frac{\partial v_t}{\partial y} \end{pmatrix}.$$

From this computation, try to think what you can conclude about what the phase flow does to the phase space (e.g., is area conserved? Will a small area become large as it carried along by the flow, or vice versa?)

Solution. In the proof of Liouville's theorem we saw that J_t was given by the formula

$$J_t = \exp\left(\int_0^t \mathrm{div}(F, G)\, ds\right)$$

which in this case evaluates to

$$J_t = e^{-\alpha t}.$$

Geometrically, the interpretation is that phase space area is not conserved (which we already knew, since area conservation is equivalent to the system being Hamiltonian), and furthermore we see that phase space area shrinks exponentially over time. The reason is that the damping term $-\alpha y$ causes energy to dissipate, and all trajectories spiral into either an asymptotically stable rest point or a limit cycle. See section 8.5 (pages 265–273) in Strogatz's book *Nonlinear Dynamics and Chaos* for a detailed study of the behavior of this system as a function of the parameters α, I.