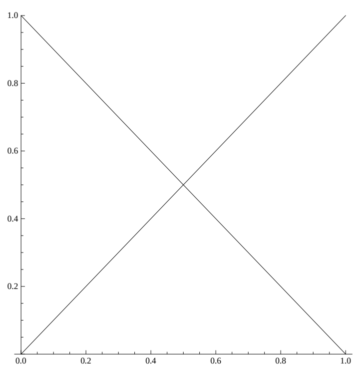
Problems

1. For each of the following maps acting on the interval $[0, 1]$, sketch their graphs, find their fixed points and determine for each fixed point whether it is asymptotically stable (a.k.a. **attracting**), asymptotically unstable (a.k.a. **repelling**), or neither:

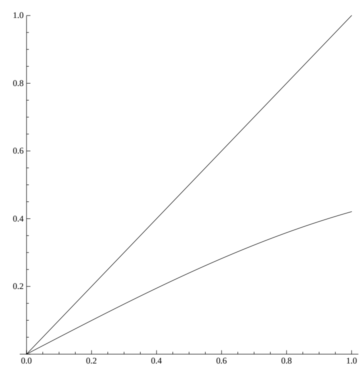
   i. $T(x) = 1 - x$ ii. $T(x) = \frac{1}{2}\sin x$ iii. $T(x) = \dfrac{e^x - 0.5}{e}$ iv. $T(x) = (2x-1)^2$

   Solution. The graphs are shown below, plotted together with $y = x$.

   i. $x = 1/2$ is a fixed point. It is neutrally stable—neither attracting nor repelling.
   ii. $x = 0$ is a stable fixed point.
   iii. There is a unique fixed point, the solution of the equation $e^x = \frac{1}{2} + ex$, which cannot be solved analytically. Its numerical value is approximately 0.3254, and by evaluating the derivative of the map we see that it is asymptotically stable.
   iv. There are two fixed points, $x = 1/4$ and $x = 1$, both repelling.



i.



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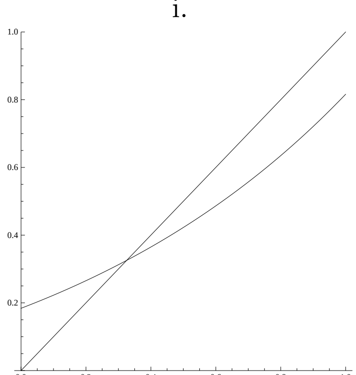
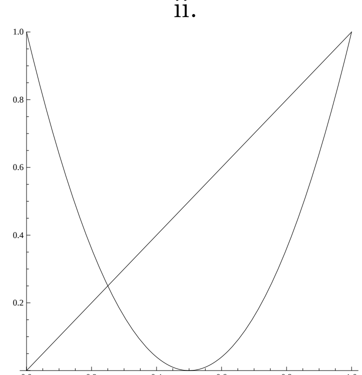


iii.



iv.

2. (a) For each of the following maps acting on $\mathbb{R}$, investigate numerically (by iterating the map using a computer or calculator) their stability behavior in the neighborhood of the fixed point $x_* = 0$. Try both negative and positive initial values and determine if the fixed point is attracting or repelling from the left and from the right (note that a mixed stability type is possible, with different behavior from different sides of approach).

   i. $T(x) = x + x^2$

   Answer. Attracting on the left, repelling on the right, since $T(x) > x$ for all $x \neq 0$.

   ii. $T(x) = x - x^2$

   Answer. Repelling on the left, attracting to the right, since $T(x) < x$ for all $x \neq 0$.

   iii. $T(x) = -x + x^2$

   Answer. Attracting from both sides (the convergence to the fixed point is oscillatory; draw a cobweb diagram or apply similar reasoning as in i.–ii. above to $T \circ T(x) = x - 2x^3 + x^4$).

   iv. $T(x) = -x - x^2$

   Answer. Attracting from both sides (same explanation as iii.).

(b) Let x_* be a fixed point of an interval map T. Denote $\lambda = T'(x_*), \mu = T''(x_*)$, so that the second-order Taylor expansion of T around x_* has the form

$$T(x) = x_* + \lambda(x - x_*) + \tfrac{1}{2}\mu(x - x_*)^2 + O((x - x_*)^3).$$

From the answer to part (a) above, formulate a guess as to how the stability type of a fixed point x_* can be determined in the boundary case $\lambda = \pm 1$, under the assumption that $\mu \neq 0$.

Solution. The behavior of the examples described in part (a) is prototypical for functions with a second-order Taylor expansion of the given form. In general we have three cases:

   i. $\lambda = 1$, $\mu > 0$: the fixed point is attracting on the left, repelling on the right.

   ii. $\lambda = 1$, $\mu < 0$: the fixed point is repelling on the left, attracting on the right.

   iii. $\lambda = -1$, $\mu \neq 0$: the fixed point is attracting on both sides (i.e., it is asymptotically stable).

3. In 19th-century Europe, family names were passed on only to male descendants. The Galton family[1], a family of noblemen, had a tradition that each male family member should have precisely 3 children. That means that if in the nth generation there were g_n male Galton family descendants, the $(n+1)$th generation will have a *random* number g_{n+1} of male descendants, since each nth generation male will have anywhere between 0 and 3 male offspring with different probabilities (thus, $(g_n)_{n=0}^{\infty}$ is an example of a *random dynamical system* or *random process*, a type of mathematical object we will not study in this course).

   Denote by $P_{n,k}$ the probability that in the nth generation there were exactly k male Galton family descendants, assuming the initial condition $x_0 = 1$ (i.e., the entire family was descended from a single "patriarch" at generation 0). It can be shown using elementary probability theory that

$$P_{n,0} + P_{n,1}x + P_{n,2}x^2 + P_{n,3}x^3 + \ldots = \sum_{k=0}^{3^n} P_{n,k}x^k = (\overbrace{f \circ f \circ \ldots \circ f}^{n \text{ times}})(x) = f^n(x), \qquad (1)$$

   where $f(x)$ is the polynomial

$$f(x) = \frac{1 + 3x + 3x^2 + x^3}{8}.$$

   In words: the polynomial whose coefficients are the probabilities $P_{n,k}$ for $0 \le k \le 3^n$ describing the distribution of the number of male descendants in the nth generation is exactly the nth functional iterate of f. In particular, for $n = 1$ this is equivalent to the statement that $P_{1,0} = \frac{1}{8}, P_{1,1} = \frac{3}{8}, P_{1,2} = \frac{3}{8}, P_{1,3} = \frac{1}{8}$. These values are the easily-computed probabilities for the different numbers of boys in a family with 3 children (assuming 50% of babies are born male—in reality, for human babies the actual percentage is around 51%).

   (a) The number $P_{n,0}$ represents the probability that the family name has died out by the nth generation. Compute it for $n = 0, 1, 2$. For general n, write a formula expressing it in terms of the map f.

   Solution.

$$P_{n,0} = \sum_{k=0}^{3^n} P_{n,k}x^k \Big|_{x=0} = f^n(0).$$

   For $n = 0, 1, 2$ this gives

$$\begin{aligned} &P_{0,0} = 0, \\ &P_{1,0} = f(0) = 1/8 = 0.125, \\ &P_{2,0} = f(f(0)) = f(1/8) = 729/4096 \approx 0.178. \end{aligned}$$

[1] Historical note: the mathematical process described in this question is an important and much-studied model called the **Galton-Watson process**, named after Francis Galton and Henry Watson.

(b) Compute the limit $\lim_{n\to\infty} P_{n,0}$ (the probability that the family name will eventually die out).

Solution. By solving the equation $f(x) = x$ we find that the map f has two fixed points in $[0, 1]$, $x_1 = 1$ and $x_2 = \sqrt{5} - 2 \approx 0.236$. We also find that

$$f'(x_1) = f'(1) = 3/2, \quad f'(x_2) = \frac{9 - 3\sqrt{5}}{4} \approx 0.573.$$

Since $|f'(x_1)| > 1$ and $|f'(x_2)| < 1$, this implies that x_1 is a repelling fixed point and x_2 is an attracting fixed point. It follows that $f^n(x_0) \to x_2$ for any initial condition $x_0 \in [0, 1]$. In particular, the probability for the noblemen's family name to eventually die out is

$$\text{Prob(die out)} = \lim_{n\to\infty} P_{n,0} = \lim_{n\to\infty} f^n(0) = x_2 = \sqrt{5} - 2 \approx 23.6\%.$$

4. **Newton's method** is a technique in numerical analysis to numerically solve equations of the form $g(x) = 0$ where g is a (sufficiently well-behaved) function defined on some interval. Given g, we define an evolution map $x_{n+1} = T(x_n)$ by

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}.$$

(a) Write the evolution map associated with the equation $g(x) = x^2 - 2 = 0$.

Solution. $T(x) = x - \frac{x^2-2}{2x} = \frac{x^2+2}{2x}$.

(b) In this specific example, show that the fixed points of the map T correspond exactly to the solutions of the equation $g(x) = 0$. Generalize this to arbitrary functions g.

Solution. The equation $\frac{x^2+2}{2x} = x$ reduces to $-x^2+2 = 0$, which is the same as $g(x) = 0$. In general, by the definition of T we have $T(x) - x = -\frac{g(x)}{g'(x)}$, so the equation $T(x) = x$ is equivalent to $g(x) = 0$ for any g.

(c) In this example, show that the fixed points of T are **superstable** (a fixed point x_* is called superstable if $T'(x_*) = 0$, which means that the convergence to the fixed point is even faster than exponential). Generalize this to arbitrary g.

Solution. The derivative of the map T is

$$T'(x) = \frac{d}{dx}\left(x - \frac{g(x)}{g'(x)}\right) = 1 - \frac{g'(x)^2 - g(x)g''(x)}{g'(x)^2} = \frac{g(x)g''(x)}{g'(x)^2}.$$

If x_* is a fixed point of T, then $T(x_*) = x_*$, which we already saw is equivalent to $g(x) = 0$. In this case $T'(x_*) = \frac{g(x_*)g''(x_*)}{g'(x_*)^2} = 0$, which means that the fixed point is superstable.

(d) For the example, compute the first 6 iterations of T starting from $x_0 = 1$. Note the rapid convergence to the root $\sqrt{2}$.

Solution. Here are the answers, computed to a precision of 20 digits:

$$\begin{aligned} T(1) &= 1.5000000000000000000, \\ T^2(1) &= 1.4166666666666666667, \\ T^3(1) &= 1.4142156862745098039, \\ T^4(1) &= 1.4142135623746899106, \\ T^5(1) &= 1.4142135623730950488, \\ T^6(1) &= 1.4142135623730950488. \end{aligned}$$

5. For a given number $x_0 \in [0,1)$, the sequence $x_n = (2^n x_0 \bmod 1)$ satisfies the doubling map recurrence $x_{n+1} = D(x_n)$. Show that the sequence y_n defined from x_n by

$$y_n = \sin^2(\pi x_n)$$

satisfies the recurrence $y_{n+1} = L_4(y_n)$, where $L_4(x) = 4x(1-x)$ is the case $r = 4$ of the logistic map. In other words, in the special case $r = 4$, the logistic map recurrence can be solved explicitly in terms of the solution to the doubling map.

Solution.

$$\begin{aligned}
y_{n+1} &= \sin^2(\pi x_{n+1}) = \sin^2(\pi D(x_n)) = \sin^2(\pi(2x_n \bmod 1)) \\
&= \begin{cases} \sin^2(\pi \cdot 2x_n) & \text{if } x_n < 1/2, \\ \sin^2(\pi \cdot (2x_n - 1)) & \text{if } x_n \geq 1/2, \end{cases} \\
&= \begin{cases} \sin^2(2\pi x_n) & \text{if } x_n < 1/2, \\ (-\sin(2\pi x_n))^2 & \text{if } x_n \geq 1/2, \end{cases} \\
&= \sin^2(2\pi x_n).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
L_4(y_n) &= 4\sin^2(\pi x_n)(1 - \sin^2(\pi x_n)) = 4\sin^2(\pi x_n)\cos^2(\pi x_n) \\
&= (2\sin(\pi x_n)\cos(\pi x_n))^2 = \sin^2(2\pi x_n).
\end{aligned}$$

Since the two expressions are equal, this shows that $y_{n+1} = L_4(y_n)$.