Homework due: Friday 5/25 in class

Problems

1. For each of the following maps acting on the interval $[0,1]$, sketch their graphs, find their fixed points and determine for each fixed point whether it is asymptotically stable (a.k.a. **attracting**), asymptotically unstable (a.k.a. **repelling**), or neither:

   i. $T(x) = 1 - x$ ii. $T(x) = \frac{1}{2}\sin x$ iii. $T(x) = \dfrac{e^x - 0.5}{e}$ iv. $T(x) = (2x-1)^2$

2. (a) For each of the following maps acting on $\mathbb{R}$, investigate numerically (by iterating the map using a computer or calculator) their stability behavior in the neighborhood of the fixed point $x_* = 0$. Try both negative and positive initial values and determine if the fixed point is attracting or repelling from the left and from the right (note that a mixed stability type is possible, with different behavior from different sides of approach).

   i. $T(x) = x + x^2$
   ii. $T(x) = x - x^2$
   iii. $T(x) = -x + x^2$
   iv. $T(x) = -x - x^2$

   (b) Let x_* be a fixed point of an interval map T. Denote $\lambda = T'(x_*), \mu = T''(x_*)$, so that the second-order Taylor expansion of T around x_* has the form

   $$T(x) = x_* + \lambda(x - x_*) + \tfrac{1}{2}\mu(x - x_*)^2 + O((x - x_*)^3).$$

   From the answer to part (a) above, formulate a guess as to how the stability type of a fixed point x_* can be determined in the boundary case $\lambda = \pm 1$, under the assumption that $\mu \neq 0$.

   Hint. The answer depends on the sign of both λ and μ.

3. In 19th-century Europe, family names were passed on only to male descendants. The Galton family[1], a family of noblemen, had a tradition that each male family member should have precisely 3 children. That means that if in the nth generation there were g_n male Galton family descendants, the $(n+1)$th generation will have a *random* number g_{n+1} of male descendants, since each nth generation male will have anywhere between 0 and 3 male offspring with different probabilities (thus, $(g_n)_{n=0}^\infty$ is an example of a *random dynamical system* or *random process*, a type of mathematical object we will not study in this course).

   Denote by $P_{n,k}$ the probability that in the nth generation there were exactly k male Galton family descendants, assuming the initial condition $x_0 = 1$ (i.e., the entire family was descended from a single "patriarch" at generation 0). It can be shown using elementary probability theory that

   $$P_{n,0} + P_{n,1}x + P_{n,2}x^2 + P_{n,3}x^3 + \ldots = \sum_{k=0}^{3^n} P_{n,k}x^k = (\overbrace{f \circ f \circ \ldots \circ f}^{n \text{ times}})(x) = f^n(x), \quad (1)$$

[1] Historical note: the mathematical process described in this question is an important and much-studied model called the **Galton-Watson process**, named after Francis Galton and Henry Watson.

where $f(x)$ is the polynomial

$$f(x) = \frac{1 + 3x + 3x^2 + x^3}{8}.$$

In words: the polynomial whose coefficients are the probabilities $P_{n,k}$ for $0 \le k \le 3^n$ describing the distribution of the number of male descendants in the nth generation is exactly the nth functional iterate of f. In particular, for $n = 1$ this is equivalent to the statement that $P_{1,0} = \frac{1}{8}, P_{1,1} = \frac{3}{8}, P_{1,2} = \frac{3}{8}, P_{1,3} = \frac{1}{8}$. These values are the easily-computed probabilities for the different numbers of boys in a family with 3 children (assuming 50% of babies are born male—in reality, for human babies the actual percentage is around 51%).

(a) The number $P_{n,0}$ represents the probability that the family name has died out by the nth generation. Compute it for $n = 0, 1, 2$. For general n, write a formula expressing it in terms of the map f.

(b) Compute the limit $\lim_{n\to\infty} P_{n,0}$ (the probability that the family name will eventually die out).

Hint. This is related to the fixed points of f and their stability.

4. **Newton's method** is a technique in numerical analysis to numerically solve equations of the form $g(x) = 0$ where g is a (sufficiently well-behaved) function defined on some interval. Given g, we define an evolution map $x_{n+1} = T(x_n)$ by

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}.$$

(a) Write the evolution map associated with the equation $g(x) = x^2 - 2 = 0$.

(b) In this specific example, show that the fixed points of the map T correspond exactly to the solutions of the equation $g(x) = 0$. Generalize this to arbitrary functions g.

(c) In this example, show that the fixed points of T are **superstable** (a fixed point x_* is called superstable if $T'(x_*) = 0$, which means that the convergence to the fixed point is even faster than exponential). Generalize this to arbitrary g.

(d) For the example, compute the first 6 iterations of T starting from $x_0 = 1$. Note the rapid convergence to the root $\sqrt{2}$.

5. For a given number $x_0 \in [0, 1)$, the sequence $x_n = (2^n x_0 \bmod 1)$ satisfies the doubling map recurrence $x_{n+1} = D(x_n)$. Show that the sequence y_n defined from x_n by

$$y_n = \sin^2(\pi x_n)$$

satisfies the recurrence $y_{n+1} = L_4(y_n)$, where $L_4(x) = 4x(1 - x)$ is the case $r = 4$ of the logistic map. In other words, in the special case $r = 4$, the logistic map recurrence can be solved explicitly in terms of the solution to the doubling map.