Homework Assignment #6  Math 119B  UC Davis, Spring 2012

Homework due:  Friday 5/25 in class

Problems

1. For each of the following maps acting on the interval [0, 1], sketch their graphs, find their fixed points and determine for each fixed point whether it is asymptotically stable (a.k.a. attracting), asymptotically unstable (a.k.a. repelling), or neither:
   i. \( T(x) = 1 - x \)
   ii. \( T(x) = \frac{1}{2} \sin x \)
   iii. \( T(x) = \frac{e^x - 0.5}{e} \)
   iv. \( T(x) = (2x - 1)^2 \)

2. (a) For each of the following maps acting on \( \mathbb{R} \), investigate numerically (by iterating the map using a computer or calculator) their stability behavior in the neighborhood of the fixed point \( x^* = 0 \). Try both negative and positive initial values and determine if the fixed point is attracting or repelling from the left and from the right (note that a mixed stability type is possible, with different behavior from different sides of approach).
   i. \( T(x) = x + x^2 \)
   ii. \( T(x) = x - x^2 \)
   iii. \( T(x) = -x + x^2 \)
   iv. \( T(x) = -x - x^2 \)

(b) Let \( x^* \) be a fixed point of an interval map \( T \). Denote \( \lambda = T'(x^*), \mu = T''(x^*) \), so that the second-order Taylor expansion of \( T \) around \( x^* \) has the form
   \[ T(x) = x^* + \lambda (x - x^*) + \frac{1}{2}\mu (x - x^*)^2 + O((x - x^*)^3). \]

From the answer to part (a) above, formulate a guess as to how the stability type of a fixed point \( x^* \) can be determined in the boundary case \( \lambda = \pm 1 \), under the assumption that \( \mu \neq 0 \).

Hint. The answer depends on the sign of both \( \lambda \) and \( \mu \).

3. In 19th-century Europe, family names were passed on only to male descendants. The Galton family\(^1\), a family of noblemen, had a tradition that each male family member should have precisely 3 children. That means that if in the \( n \)th generation there were \( g_n \) male Galton family descendants, the \( (n+1) \)th generation will have a random number \( g_{n+1} \) of male descendants, since each \( nth \) generation male will have anywhere between 0 and 3 male offspring with different probabilities (thus, \( (g_n)_{n=0}^\infty \) is an example of a random dynamical system or random process, a type of mathematical object we will not study in this course).

Denote by \( P_{n,k} \) the probability that in the \( n \)th generation there were exactly \( k \) male Galton family descendants, assuming the initial condition \( x_0 = 1 \) (i.e., the entire family was descended from a single “patriarch” at generation 0). It can be shown using elementary probability theory that

\[ P_{n,0} + P_{n,1} x + P_{n,2} x^2 + P_{n,3} x^3 + \ldots = \sum_{k=0}^{3^n} P_{n,k} x^k = (f \circ f \circ \ldots \circ f)(x) = f^n(x), \quad (1) \]

\(^1\)Historical note: the mathematical process described in this question is an important and much-studied model called the Galton-Watson process, named after Francis Galton and Henry Watson.
where \( f(x) \) is the polynomial
\[
f(x) = \frac{1 + 3x + 3x^2 + x^3}{8}.
\]

In words: the polynomial whose coefficients are the probabilities \( P_{n,k} \) for \( 0 \leq k \leq 3^n \) describing the distribution of the number of male descendants in the \( n \)th generation is exactly the \( n \)th functional iterate of \( f \). In particular, for \( n = 1 \) this is equivalent to the statement that \( P_{1,0} = \frac{1}{8}, P_{1,1} = \frac{3}{8}, P_{1,2} = \frac{3}{8}, P_{1,3} = \frac{1}{8} \). These values are the easily-computed probabilities for the different numbers of boys in a family with 3 children (assuming 50% of babies are born male—in reality, for human babies the actual percentage is around 51%).

(a) The number \( P_{n,0} \) represents the probability that the family name has died out by the \( n \)th generation. Compute it for \( n = 0, 1, 2 \). For general \( n \), write a formula expressing it in terms of the map \( f \).

(b) Compute the limit \( \lim_{n \to \infty} P_{n,0} \) (the probability that the family name will eventually die out).

**Hint.** This is related to the fixed points of \( f \) and their stability.

4. **Newton’s method** is a technique in numerical analysis to numerically solve equations of the form \( g(x) = 0 \) where \( g \) is a (sufficiently well-behaved) function defined on some interval. Given \( g \), we define an evolution map \( x_{n+1} = T(x_n) \) by

\[
x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}.
\]

(a) Write the evolution map associated with the equation \( g(x) = x^2 - 2 = 0 \).

(b) In this specific example, show that the fixed points of the map \( T \) correspond exactly to the solutions of the equation \( g(x) = 0 \). Generalize this to arbitrary functions \( g \).

(c) In this example, show that the fixed points of \( T \) are superstable (a fixed point \( x^* \) is called superstable if \( T'(x^*) = 0 \), which means that the convergence to the fixed point is even faster than exponential). Generalize this to arbitrary \( g \).

(d) For the example, compute the first 6 iterations of \( T \) starting from \( x_0 = 1 \). Note the rapid convergence to the root \( \sqrt{2} \).

5. For a given number \( x_0 \in [0,1) \), the sequence \( x_n = (2^n x_0 \mod 1) \) satisfies the doubling map recurrence \( x_{n+1} = D(x_n) \). Show that the sequence \( y_n \) defined from \( x_n \) by

\[
y_n = \sin^2(\pi x_n)
\]

satisfies the recurrence \( y_{n+1} = L_4(y_n) \), where \( L_4(x) = 4x(1-x) \) is the case \( r = 4 \) of the logistic map. In other words, in the special case \( r = 4 \), the logistic map recurrence can be solved explicitly in terms of the solution to the doubling map.