1. A fixed point x_* of an interval map T is called **superstable** if $T'(x_*) = 0$. Find the value of $0 < r \le 4$ for which the logistic map L_r has a superstable fixed point.

   Solution. We know that the fixed point x_* of L_r is given by $x_* = 0$ for $r \le 1$ and $x_* = \frac{r-1}{r}$ for $1 \le r \le 4$. Furthermore, the derivative of the logistic map is

$$L_r'(x) = r(1 - 2x).$$

   Thus, $L_r'(x_*) = r$ for $0 < r \le 1$, which means that in this range there are no superstable fixed points, and

$$L_r'(x_*) = r(1 - 2(r-1)/r) = r - 2(r-1) = 2 - r.$$

   This is equal to 0 when $r = 2$. The superstable fixed point for this r is therefore $x_* = \frac{1}{2}$.

2. Let (p, q) be a 2-cycle of the logistic map L_r, that is, a pair of numbers such that

$$p = L_r(q), \quad q = L_r(p).$$

Assume that the 2-cycle is superstable, i.e., that p and q are both superstable fixed points of the iterated map $L_r \circ L_r$.

(a) Show that in this case one of p, q must be equal to $1/2$.

Solution. Equating the derivative $(L_r \circ L_r)'(p)$ to 0 gives:

$$(L_r \circ L_r)'(p) = L_r'(p)L_r'(L_r(p)) = L_r'(p)L_r'(q) = 0,$$

so either $L_r'(p) = 0$ or $L_r'(q) = 0$. Since $L_r'(x) = 1 - 2x = 0$ only for $x = 1/2$, this shows that $p = 1/2$ or $q = 1/2$.

(b) Find the value of r that makes such a superstable 2-cycle possible, and find (p, q) in this case.

Solution. Assume without loss of generality that $p = 1/2$ (otherwise we can switch the roles of p and q). Then $q = L_r(p) = L_r(1/2) = r/4$, and

$$1/2 = p = L_r(q) = L_r(r/4) = r(r/4)(1 - r/4) = \frac{r^2(4 - r)}{16}.$$

This gives an equation for r, which simplifies to

$$r^3 - 4r^2 + 8 = 0.$$

This cubic equation has an obvious solution $r = 2$ (which can be predicted from problem 1 above, since in that case $p = q = 1/2$ is a superstable fixed point repeated twice (which is not technically a 2-cycle but still produces a pair p, q satisfying the same relations). Using this, we can factor the equation into the form

$$(r - 2)(r^2 - 2r - 4) = 0,$$

and therefore deduce the other two solutions $r_{1,2} = 1 \pm \sqrt{5}$, of which $r = 1 + \sqrt{5}$ is in the range $[1, 4]$ we are considering. To summarize, we have found the superstable 2-cycle:

$$r = 1 + \sqrt{5} = 3.236...,$$
$$(p, q) = (1/2, L_r(1/2)) = (1/2, (1 + \sqrt{5})/4) = (0.5, 0.809...).$$

3. As the parameter r of the logistic map L_r is gradually increased, let r_k be the value of r at which the kth flip bifurcation occurs, giving rise to a 2^k-cycle. The first few values of r_k are given in the following table:

k		r_k
1	(2-cycle is born)	3
2	(4-cycle)	$3.44949... = 1+\sqrt{6}$
3	(8-cycle)	3.54409...
4	(16-cycle)	3.568759...
$\vdots$		
∞	(onset of chaos)	3.569946...

The sequence r_k is known to approach its limit r_∞ at a roughly geometric rate, with an exponent

$$\delta = \lim_{k\to\infty} \frac{r_k - r_{k-1}}{r_{k+1} - r_k} = 4.669201...$$

representing the factor by which r_k approaches r_∞ with each successive iteration (this number is known as **Feigenbaum's constant**, after its discoverer). The rapid succession of bifurcations leading to the onset of chaos at the value r_∞ has been nicely explained in terms of a process called **renormalization**, described in section 10.7 in Strogatz's book *Nonlinear Dynamics and Chaos*. In this problem we use this theory to derive approximate expressions for r_k, r_∞ and δ.

(a) A simplified version of the renormalization analysis (see the section titled "Renormalization for Pedestrians" on pages 384–387 in Strogatz's book) suggests that r_k are approximated by a sequence $(q_k)_{k=1}^\infty$ defined in terms of the recurrence

$$q_{k+1} = T(q_k) \qquad (k = 1, 2, \ldots),$$

with the initial condition $q_1 = r_1 = 3$, where T is the map $T(x) = 1 + \sqrt{3+x}$. Compute the first few q_k for $k = 1, 2, 3, 4$ and compare them to the values of r_k in the table above.

Solution.

k	q_k	r_k
1	3	3
2	$3.44949... = 1+\sqrt{6}$	$3.44949... = 1+\sqrt{6}$
3	$3.53958... = 1+\sqrt{4+\sqrt{6}}$	3.54409...
4	$3.55726... = 1+\sqrt{4+\sqrt{4+\sqrt{6}}}$	3.568759...

(b) Analyze the recurrence defining q_k to show that q_k converges to a limit q_∞ as $k \to \infty$, find a formula for q_∞ and compute its numerical value. Compare the value you obtained to the value of r_∞ given above.

Solution. Since the sequence $(q_k)_{k=1}^{\infty}$ is computed by iterations of the map T, we expect that it may converge to a fixed point of the map. The fixed points of this map are solutions of the equation

$$x = 1 + \sqrt{3 + x},$$

which simplifies to give the quadratic equation

$$x^2 - 3x - 2 = 0,$$

whose positive solution is $x_* = \frac{3+\sqrt{17}}{2} = 3.56155...$ (the other solution is negative and therefore does not solve the original equation $x = 1 + \sqrt{3 + x}$). This fixed point is seen to be asymptotically stable, since (we find after a short computation)

$$T'(x_*) = \frac{1}{1 + \sqrt{17}} = 0.1952...$$

It follows that the sequence q_k converges to the limiting value $q_\infty = \frac{3+\sqrt{17}}{2}$. This deviates from the value r_∞ by 0.0072..., a relative error of about one fifth of a percent.

(c) In the approximate model, the analogous quantity to Feigenbaum's constant δ is the reciprocal of the convergence exponent $\lambda = T'(x_*)$ associated with an asymptotically stable fixed point of the map. This leads to the approximate formula

$$\delta_{\text{approx}} = \left(\frac{dq_{k+1}}{dq_k}_{\big|q_k = q_\infty} \right)^{-1} = \frac{1}{T'(q_\infty)}.$$

Compute the numerical value of this number and compare it to the precise value for δ given above.

Solution. The reciprocal of δ_{approx} was computed above as part of the stability analysis of the fixed point. We have

$$\delta_{\text{approx}} = 1 + \sqrt{17} = 5.123...,$$

a deviation of around 10% from Feigenbaum's constant.

4. Let $T : I \to I$ be an interval map. A probability measure $P(A) = \int_A f(x)\,dx$ is invariant under T if the equation $P(T^{-1}(A)) = P(A)$ holds for any (reasonable) set $A \subset [0,1]$. By an analysis similar to the one we did in class for the logistic map L_4 it is possible to show that an equivalent condition for P to be an invariant measure is that the density function f satisfies the equation

$$f(x) = \sum_{y \in T^{-1}(x)} \frac{f(y)}{|T'(y)|},$$

where the sum is over all the pre-images of x under T, i.e., all the solutions of the equation $T(y) = x$. For example, in the case of the logistic map L_4 this becomes the identity

$$f(x) = \frac{f(\lambda_1(x))}{|L_4'(\lambda_1(x))|} + \frac{f(\lambda_2(x))}{|L_4'(\lambda_2(x))|},$$

where $f(x) = \frac{1}{\pi\sqrt{x(1-x)}}$ (see Example 29 on pages 56–57 in the lecture notes).

For each of the following pairs consisting of an interval map and a density function, verify that the above identity holds (and therefore that the associated measure is invariant under the map).

(a) $f(x) = 1$ with the doubling map $D(x) = 2x \bmod 1$ on $[0,1)$.

Solution. The pre-images of any point $x \in [0,1]$ are $y_1 = x/2$ and $y_2 = x/2 + 1/2$, and $D'(y) = 2$ for all y, so we have

$$\sum_{y \in D^{-1}(x)} \frac{f(y)}{|D'(y)|} = \frac{1}{2} + \frac{1}{2} = 1 = f(x).$$

(b) $f(x) = \frac{1}{x}$ with the map $A(x) = \begin{cases} \frac{x}{1-x} & \text{if } 0 \le x \le 1/2, \\ \frac{1-x}{x} & \text{if } 1/2 \le x \le 1. \end{cases}$ on $[0,1]$.

Solution. By plotting the map we see that each point $x \in [0,1]$ (except $x = 1/2$, where in any case A is not differentiable) has two pre-images $y_1 = \frac{x}{1+x} \in [0,1/2)$ and $y_2 = 1/(1+x) \in (1/2,1]$. We also have $A'(y_1) = 1/(1-y_1)^2$, $A'(y_2) = -1/y_2^2$. To verify the identity, we compute:

$$\sum_{y \in A^{-1}(x)} \frac{f(y)}{|A'(y)|} = \frac{f(y_1)}{1/(1-y_1)^2} + \frac{f(y_2)}{1/y_2^2} = \frac{(1-y_1)^2}{y_1} + \frac{y_2^2}{y_2}$$
$$= \frac{1}{x(1+x)} + \frac{1}{1+x} = \frac{1}{x} = f(x).$$

(c) $f(x) = 1$ with the map $B(x) = x - \frac{1}{x}$ (**Boole's map**) on $\mathbb{R}$.

Solution. The solutions in y to the equation $B(y) = x$ are $y_{1,2} = \frac{x \pm \sqrt{x^2+4}}{2}$, so (since $f(x) = 1$) the right hand-side of the identity is

$$\frac{1}{1 + y_1^{-2}} + \frac{1}{1 + y_2^{-2}}.$$

With a bit of algebra it is easy to verify that this is equal to $f(x) = 1$.