1. A fixed point $x_*$ of an interval map $T$ is called superstable if $T'(x_*) = 0$. Find the value of $0 < r \leq 4$ for which the logistic map $L_r$ has a superstable fixed point.

**Solution.** We know that the fixed point $x_*$ of $L_r$ is given by $x_* = 0$ for $r \leq 1$ and $x_* = \frac{r-1}{r}$ for $1 \leq r \leq 4$. Furthermore, the derivative of the logistic map is

$$L'_r(x) = r(1 - 2x).$$

Thus, $L'_r(x_*) = r$ for $0 < r \leq 1$, which means that in this range there are no superstable fixed points, and

$$L'_r(x_*) = r(1 - 2(r-1)/r) = r - 2(r-1) = 2 - r.$$

This is equal to 0 when $r = 2$. The superstable fixed point for this $r$ is therefore $x_* = \frac{1}{2}$. 
2. Let \((p, q)\) be a 2-cycle of the logistic map \(L_r\), that is, a pair of numbers such that
\[
p = L_r(q), \quad q = L_r(p).
\]
Assume that the 2-cycle is superstable, i.e., that \(p\) and \(q\) are both superstable fixed points of the iterated map \(L_r \circ L_r\).

(a) Show that in this case one of \(p, q\) must be equal to \(1/2\).

**Solution.** Equating the derivative \((L_r \circ L_r)'(p)\) to 0 gives:
\[
(L_r \circ L_r)'(p) = L'_r(p)L'_r(L_r(p)) = L'_r(p)L'_r(q) = 0,
\]
so either \(L'_r(p) = 0\) or \(L'_r(q) = 0\). Since \(L'_r(x) = 1 - 2x = 0\) only for \(x = 1/2\), this shows that \(p = 1/2\) or \(q = 1/2\).

(b) Find the value of \(r\) that makes such a superstable 2-cycle possible, and find \((p, q)\) in this case.

**Solution.** Assume without loss of generality that \(p = 1/2\) (otherwise we can switch the roles of \(p\) and \(q\)). Then \(q = L_r(p) = L_r(1/2) = r/4\), and
\[
1/2 = p = L_r(q) = L_r(r/4) = r(r/4)(1 - r/4) = \frac{r^2(4 - r)}{16}.
\]
This gives an equation for \(r\), which simplifies to
\[
r^3 - 4r^2 + 8 = 0.
\]
This cubic equation has an obvious solution \(r = 2\) (which can be predicted from problem 1 above, since in that case \(p = q = 1/2\) is a superstable fixed point repeated twice (which is not technically a 2-cycle but still produces a pair \(p, q\) satisfying the same relations). Using this, we can factor the equation into the form
\[
(r - 2)(r^2 - 2r - 4) = 0,
\]
and therefore deduce the other two solutions \(r_{1,2} = 1 \pm \sqrt{5}\), of which \(r = 1 + \sqrt{5}\) is in the range \([1, 4]\) we are considering. To summarize, we have found the superstable 2-cycle:
\[
r = 1 + \sqrt{5} = 3.236...,\]
\[
(p, q) = (1/2, L_r(1/2)) = (1/2, (1 + \sqrt{5})/4) = (0.5, 0.809...).
\]
3. As the parameter $r$ of the logistic map $L_r$ is gradually increased, let $r_k$ be the value of $r$ at which the $k$th flip bifurcation occurs, giving rise to a $2^k$-cycle. The first few values of $r_k$ are given in the following table:

<table>
<thead>
<tr>
<th>$k$</th>
<th>$r_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(2-cycle is born) 3</td>
</tr>
<tr>
<td>2</td>
<td>(4-cycle) 3.44949... = 1 + $\sqrt{6}$</td>
</tr>
<tr>
<td>3</td>
<td>(8-cycle) 3.54409...</td>
</tr>
<tr>
<td>4</td>
<td>(16-cycle) 3.568759...</td>
</tr>
<tr>
<td>$\infty$ (onset of chaos)</td>
<td>3.569946...</td>
</tr>
</tbody>
</table>

The sequence $r_k$ is known to approach its limit $r_\infty$ at a roughly geometric rate, with an exponent

$$
\delta = \lim_{k \to \infty} \frac{r_k - r_{k-1}}{r_{k+1} - r_k} = 4.669201...
$$

representing the factor by which $r_k$ approaches $r_\infty$ with each successive iteration (this number is known as Feigenbaum’s constant, after its discoverer). The rapid succession of bifurcations leading to the onset of chaos at the value $r_\infty$ has been nicely explained in terms of a process called renormalization, described in section 10.7 in Strogatz’s book *Nonlinear Dynamics and Chaos*. In this problem we use this theory to derive approximate expressions for $r_k$, $r_\infty$ and $\delta$.

(a) A simplified version of the renormalization analysis (see the section titled “Renormalization for Pedestrians” on pages 384–387 in Strogatz’s book) suggests that $r_k$ are approximated by a sequence $(q_k)_{k=1}^\infty$ defined in terms of the recurrence

$$
q_{k+1} = T(q_k) \quad (k = 1, 2, \ldots),
$$

with the initial condition $q_1 = r_1 = 3$, where $T$ is the map $T(x) = 1 + \sqrt{3 + x}$. Compute the first few $q_k$ for $k = 1, 2, 3, 4$ and compare them to the values of $r_k$ in the table above.

**Solution.**

<table>
<thead>
<tr>
<th>$k$</th>
<th>$q_k$</th>
<th>$r_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>3.44949... = 1 + $\sqrt{6}$</td>
<td>3.44949... = 1 + $\sqrt{6}$</td>
</tr>
<tr>
<td>3</td>
<td>3.53958... = 1 + $\sqrt{4 + \sqrt{6}}$</td>
<td>3.54409...</td>
</tr>
<tr>
<td>4</td>
<td>3.55726... = 1 + $\sqrt{4 + \sqrt{4 + \sqrt{6}}}$</td>
<td>3.568759...</td>
</tr>
</tbody>
</table>

(b) Analyze the recurrence defining $q_k$ to show that $q_k$ converges to a limit $q_\infty$ as $k \to \infty$, find a formula for $q_\infty$ and compute its numerical value. Compare the value you obtained to the value of $r_\infty$ given above.
Solution. Since the sequence \((q_k)_{k=1}^\infty\) is computed by iterations of the map \(T\), we expect that it may converge to a fixed point of the map. The fixed points of this map are solutions of the equation
\[
x = 1 + \sqrt{3 + x},
\]
which simplifies to give the quadratic equation
\[
x^2 - 3x - 2 = 0,
\]
whose positive solution is \(x_* = \frac{3 + \sqrt{17}}{2} = 3.56155...\) (the other solution is negative and therefore does not solve the original equation \(x = 1 + \sqrt{3 + x}\)). This fixed point is seen to be asymptotically stable, since (we find after a short computation)
\[
T'(x_*) = \frac{1}{1 + \sqrt{17}} = 0.1952...
\]

It follows that the sequence \(q_k\) converges to the limiting value \(q_\infty = \frac{3 + \sqrt{17}}{2}\). This deviates from the value \(r_\infty\) by 0.0072..., a relative error of about one fifth of a percent.

(c) In the approximate model, the analogous quantity to Feigenbaum’s constant \(\delta\) is the reciprocal of the convergence exponent \(\lambda = T'(x_*)\) associated with an asymptotically stable fixed point of the map. This leads to the approximate formula
\[
\delta_{\text{approx}} = \left( \frac{dq_{k+1}}{dq_k} \bigg|_{q_k=q_\infty} \right)^{-1} = \frac{1}{T'(q_\infty)}.
\]
Compute the numerical value of this number and compare it to the precise value for \(\delta\) given above.

Solution. The reciprocal of \(\delta_{\text{approx}}\) was computed above as part of the stability analysis of the fixed point. We have
\[
\delta_{\text{approx}} = 1 + \sqrt{17} = 5.123...,\]
a deviation of around 10% from Feigenbaum’s constant.
4. Let $T : I \rightarrow I$ be an interval map. A probability measure $P(A) = \int_A f(x) \, dx$ is invariant under $T$ if the equation $P(T^{-1}(A)) = P(A)$ holds for any (reasonable) set $A \subset [0, 1]$. By an analysis similar to the one we did in class for the logistic map $L_4$ it is possible to show that an equivalent condition for $P$ to be an invariant measure is that the density function $f$ satisfies the equation

$$f(x) = \sum_{y \in T^{-1}(x)} \frac{f(y)}{|T'(y)|},$$

where the sum is over all the pre-images of $x$ under $T$, i.e., all the solutions of the equation $T(y) = x$. For example, in the case of the logistic map $L_4$ this becomes the identity

$$f(x) = \frac{f(\lambda_1(x))}{|L'_4(\lambda_1(x))|} + \frac{f(\lambda_2(x))}{|L'_4(\lambda_2(x))|},$$

where $f(x) = \frac{1}{\pi \sqrt{x(1-x)}}$ (see Example 29 on pages 56–57 in the lecture notes).

For each of the following pairs consisting of an interval map and a density function, verify that the above identity holds (and therefore that the associated measure is invariant under the map).

(a) $f(x) = 1$ with the doubling map $D(x) = 2x \mod 1$ on $[0, 1]$.

**Solution.** The pre-images of any point $x \in [0, 1]$ are $y_1 = x/2$ and $y_2 = x/2 + 1/2$, and $D'(y) = 2$ for all $y$, so we have

$$\sum_{y \in D^{-1}(x)} \frac{f(y)}{|D'(y)|} = \frac{1}{2} + \frac{1}{2} = 1 = f(x).$$

(b) $f(x) = \frac{1}{x}$ with the map $A(x) = \begin{cases} \frac{x}{1-x} & \text{if } 0 \leq x \leq 1/2, \\ \frac{1-x}{x} & \text{if } 1/2 \leq x \leq 1. \end{cases}$ on $[0, 1]$.

**Solution.** By plotting the map we see that each point $x \in [0, 1]$ (except $x = 1/2$, where in any case $A$ is not differentiable) has two pre-images $y_1 = \frac{x}{1-x} \in [0, 1/2)$ and $y_2 = 1/(1 + x) \in (1/2, 1]$. We also have $A'(y_1) = 1/(1 - y_1)^2$, $A'(y_2) = -1/y_2^2$. To verify the identity, we compute:

$$\sum_{y \in A^{-1}(x)} \frac{f(y)}{|A'(y)|} = \frac{f(y_1)}{1/(1 - y_1)^2} + \frac{f(y_2)}{1/y_2^2} = \frac{(1 - y_1)^2}{y_1} + \frac{y_2^2}{y_2} = \frac{1}{x(1 + x)} + \frac{1}{1 + x} = \frac{1}{x} = f(x).$$

(c) $f(x) = 1$ with the map $B(x) = x - \frac{1}{x}$ (Boole’s map) on $\mathbb{R}$. 

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**Solution.** The solutions in $y$ to the equation $B(y) = x$ are $y_{1,2} = \frac{x \pm \sqrt{x^2 + 4}}{2}$, so (since $f(x) = 1$) the right hand-side of the identity is

$$\frac{1}{1 + y_1^{-2}} + \frac{1}{1 + y_2^{-2}}.$$ 

With a bit of algebra it is easy to verify that this is equal to $f(x) = 1$. 