Homework due: Friday 6/1 in class

Problems

1. A fixed point x_* of an interval map T is called **superstable** if $T'(x_*) = 0$. Find the value of $0 < r \leq 4$ for which the logistic map L_r has a superstable fixed point.

2. Let (p, q) be a 2-cycle of the logistic map L_r, that is, a pair of numbers such that

$$p = L_r(q), \quad q = L_r(p).$$

Assume that the 2-cycle is superstable, i.e., that p and q are both superstable fixed points of the iterated map $L_r \circ L_r$.

   (a) Show that in this case one of p, q must be equal to $1/2$.

   (b) Find the value of r that makes such a superstable 2-cycle possible, and find (p, q) in this case.

3. As the parameter r of the logistic map L_r is gradually increased, let r_k be the value of r at which the kth flip bifurcation occurs, giving rise to a 2^k-cycle. The first few values of r_k are given in the following table:

k		r_k
1	(2-cycle is born)	3
2	(4-cycle)	$3.44949... = 1 + \sqrt{6}$
3	(8-cycle)	3.54409...
4	(16-cycle)	3.568759...
	$\vdots$	
∞	(onset of chaos)	3.569946...

The sequence r_k is known to approach its limit r_∞ at a roughly geometric rate, with an exponent

$$\delta = \lim_{k\to\infty} \frac{r_k - r_{k-1}}{r_{k+1} - r_k} = 4.669201...$$

representing the factor by which r_k approaches r_∞ with each successive iteration (this number is known as **Feigenbaum's constant**, after its discoverer). The rapid succession of bifurcations leading to the onset of chaos at the value r_∞ has been nicely explained in terms of a process called **renormalization**, described in section 10.7 in Strogatz's book *Nonlinear Dynamics and Chaos*. In this problem we use this theory to derive approximate expressions for r_k, r_∞ and δ.

   (a) A simplified version of the renormalization analysis (see the section titled "Renormalization for Pedestrians" on pages 384–387 in Strogatz's book) suggests that r_k are approximated by a sequence $(q_k)_{k=1}^{\infty}$ defined in terms of the recurrence

$$q_{k+1} = T(q_k) \qquad (k = 1, 2, \ldots),$$

with the initial condition $q_1 = r_1 = 3$, where T is the map $T(x) = 1 + \sqrt{3 + x}$. Compute the first few q_k for $k = 1, 2, 3, 4$ and compare them to the values of r_k in the table above.

(b) Analyze the recurrence defining q_k to show that q_k converges to a limit q_∞ as $k \to \infty$, find a formula for q_∞ and compute its numerical value. Compare the value you obtained to the value of r_∞ given above.

(c) In the approximate model, the analogous quantity to Feigenbaum's constant δ is the reciprocal of the convergence exponent $\lambda = T'(x_*)$ associated with an asymptotically stable fixed point of the map. This leads to the approximate formula

$$\delta_{\text{approx}} = \left(\frac{dq_{k+1}}{dq_k}\Big|_{q_k = q_\infty} \right)^{-1} = \frac{1}{T'(q_\infty)}.$$

Compute the numerical value of this number and compare it to the precise value for δ given above.

4. Let $T : I \to I$ be an interval map. A probability measure $P(A) = \int_A f(x)\,dx$ is invariant under T if the equation $P(T^{-1}(A)) = P(A)$ holds for any (reasonable) set $A \subset [0,1]$. By an analysis similar to the one we did in class for the logistic map L_4 it is possible to show that an equivalent condition for P to be an invariant measure is that the density function f satisfies the equation

$$f(x) = \sum_{y \in T^{-1}(x)} \frac{f(y)}{|T'(y)|},$$

where the sum is over all the pre-images of x under T, i.e., all the solutions of the equation $T(y) = x$. For example, in the case of the logistic map L_4 this becomes the identity

$$f(x) = \frac{f(\lambda_1(x))}{|L_4'(\lambda_1(x))|} + \frac{f(\lambda_2(x))}{|L_4'(\lambda_2(x))|},$$

where $f(x) = \frac{1}{\pi\sqrt{x(1-x)}}$ (see Example 29 on pages 56–57 in the lecture notes).

For each of the following pairs consisting of an interval map and a density function, verify that the above identity holds (and therefore that the associated measure is invariant under the map).

(a) $f(x) = 1$ with the doubling map $D(x) = 2x \bmod 1$ on $[0,1)$.

(b) $f(x) = \frac{1}{x}$ with the map $A(x) = \begin{cases} \frac{x}{1-x} & \text{if } 0 \le x \le 1/2, \\ \frac{1-x}{x} & \text{if } 1/2 \le x \le 1. \end{cases}$ on $[0,1]$.

(This **additive Gauss map** is related to the Euclidean algorithm and to continued fraction expansions in number theory.)

(c) $f(x) = 1$ with the map $B(x) = x - \frac{1}{x}$ (**Boole's map**) on $\mathbb{R}$.

Hint. For part (b) it is recommended to plot the map G. Note that in parts (b) and (c) the density function is not integrable, which means that the associated measure is not strictly speaking a probability measure, but the concept of an invariant measure can still be defined.