1. A child standing on a swing bends her knees up and down in a periodic motion. This causes a slight change in the resonant frequency of the swing. In the approximation of small amplitude oscillations, the equation of motion for this system is

$$\ddot{x} = -(\omega^2 \pm \epsilon^2)x = -\omega_\pm^2 x$$

where ϵ is a small number and we use the notation

$$\pm = \begin{cases} +1 & \text{if } \sin(\omega t) > 0, \\ -1 & \text{if } \sin(\omega t) < 0, \end{cases}$$

$$\omega_+ = \sqrt{\omega^2 + \epsilon^2},$$
$$\omega_- = \sqrt{\omega^2 - \epsilon^2}.$$

Note that the frequency of the knee-bending is chosen to coincide with the resonant frequency of the pendulum. The goal of this problem is to show that this causes the rest point at $x = \dot{x} = 0$ to become unstable—a phenomenon known as **parametric resonance** (that children everywhere are grateful for, since it enables them to swing on a swing without the assistance of a parent)[1].

(a) Write an equivalent form of the system as a planar first-order system.

Solution. $\dot{x} = y, \ \dot{y} = -\omega_\pm^2 x.$

(b) Use reasoning similar to our analysis of the inverted pendulum with an oscillating base to find 2×2 matrices S_+, S_- such that the criterion for stability of the system at the rest point $x = \dot{x} = 0$ can be written as $|\operatorname{tr}(P)| < 2$, where $P = S_- S_+$.

Solution. Denote $T = \pi/\omega$. The square wave function $\pm$ changes sign every T units of time. During each interval of length T where the sign remains constant, the evolution of the equation is that of a linear vector equation with constant coefficients, which as we know can be solved using matrix exponentials. Thus, the matrix of evolution for a full period $2T$ can be written as a product $S_- S_+$ where S_- and S_+ are computed as matrix exponentials, as follows:

$$S_- = \exp\left(T\begin{pmatrix} 0 & 1 \\ -\omega_-^2 & 0 \end{pmatrix}\right) = \begin{pmatrix} \cos(\omega_- T) & \frac{1}{\omega_-}\sin(\omega_- T) \\ -\omega_- \sin(\omega_- T) & \cos(\omega_- T) \end{pmatrix}$$
$$= \begin{pmatrix} \cos(\frac{\pi\omega_-}{\omega}) & \frac{1}{\omega_-}\sin(\frac{\pi\omega_-}{\omega}) \\ -\omega_- \sin(\frac{\pi\omega_-}{\omega}) & \cos(\frac{\pi\omega_-}{\omega}) \end{pmatrix},$$

$$S_+ = \exp\left(T\begin{pmatrix} 0 & 1 \\ -\omega_+^2 & 0 \end{pmatrix}\right) = \begin{pmatrix} \cos(\omega_+ T) & \frac{1}{\omega_+}\sin(\omega_+ T) \\ -\omega_+ \sin(\omega_+ T) & \cos(\omega_+ T) \end{pmatrix}$$
$$= \begin{pmatrix} \cos(\frac{\pi\omega_+}{\omega}) & \frac{1}{\omega_+}\sin(\frac{\pi\omega_+}{\omega}) \\ -\omega_+ \sin(\frac{\pi\omega_+}{\omega}) & \cos(\frac{\pi\omega_+}{\omega}) \end{pmatrix}.$$

[1] See the Wikipedia article http://en.wikipedia.org/wiki/Parametric_oscillator.

As we proved in class, the criterion for stability of the motion near $\mathbf{x} = \mathbf{0}$ is that the matrix P should satisfy $|\operatorname{tr}(P)| < 2$.

(c) Deduce from part (b) that the condition for stability is

$$\left| 2\cos\left(\frac{\pi\omega_+}{\omega}\right)\cos\left(\frac{\pi\omega_-}{\omega}\right) - \left(\frac{\omega_+}{\omega_-} + \frac{\omega_-}{\omega_+}\right)\sin\left(\frac{\pi\omega_+}{\omega}\right)\sin\left(\frac{\pi\omega_-}{\omega}\right)\right| < 2$$

Solution. The expression on the left-hand side is $|\operatorname{tr}(P)|$.

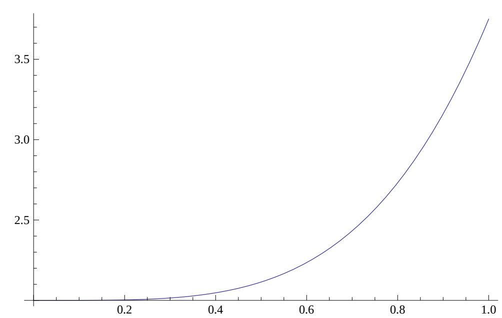
(d) Define a new variable $s = \epsilon^2/\omega^2$, and show that the condition above translates to checking that $|F(s)| < 2$, where

$$\begin{aligned} F(s) = {} & 2\cos\left(\pi\sqrt{1+s}\right)\cos\left(\pi\sqrt{1-s}\right) \\ & - \left(\frac{\sqrt{1+s}}{\sqrt{1-s}} + \frac{\sqrt{1-s}}{\sqrt{1+s}}\right)\sin\left(\pi\sqrt{1+s}\right)\sin\left(\pi\sqrt{1-s}\right). \end{aligned}$$

Solution. With the substitution $s = \epsilon^2/\omega^2$ we have $F(s) = \operatorname{tr}(P)$.

(e) Use a computer or graphing calculator to plot the graph of $F(s)$, and verify that the inequality $|F(s)| > 2$ holds for all $0 < s < 1$. Conclude that "parametrically resonated swings" are unstable (and therefore fun!).

Solution. Here is the graph of $f(s)$:



Note that the graph is very flat near $s = 0$. Magnifying the plot in the neighborhood of $s = 0$ still suggests that $f(s) > 2$ for all $s > 0$. It is not hard to check that the Taylor expansion of $f(s)$ at $s = 0$ is $f(s) = 2 + \frac{3\pi^2}{16}s^4 + O(s^6)$.

2. In the two-dimensional phase portrait of the switching control scheme

$$\dot{x} = y,$$
$$\dot{y} = -\operatorname{sgn}(x + by)$$

that arises in connection with the electromagnetic levitation problem (where b is a positive numerical parameter), the system will go into a **sliding motion** (a.k.a. **chattering**) phase after reaching the line segment AB of the switching line $x+by = 0$ shown in Figure 1(a) below. The sliding motion is characterized by the property that the vector field of the equation on both sides of the switching line pushes the particle back towards the line.

(a) Find the coordinates of the endpoints A and B of the sliding motion segment.

Hint. See Figure 1(b) and its caption below.

Solution. From the hint, we want to find the point $B = (x, y)$ where one of the curves $x = a + \frac{1}{2}y^2$ (for some value of a) becomes tangent to the line $x + by = 0$. For a given value of a, substituting the value $x = -by$ in the equation $x = a+\frac{1}{2}y^2$ gives the quadratic equation

$$\tfrac{1}{2}y^2 + by + a = 0,$$

whose solutions are

$$y_{1,2} = -b \pm \sqrt{b^2 - 2a}.$$

The value of a for which the tangency condition is satisfied is precisely the one for which the two roots coincide: $y_1 = y_2$, and in this case the solution is $y = y_1 = y_2 = -b$ and therefore $x = b^2$.

To summarize, we have found that the coordinates of the point B are given by

$$B = (b^2, -b),$$

and clearly the point A is the symmetric point

$$A = (-b^2, b).$$

(b) If we start the system at a point $(x_0, y_0) = (-by_0, y_0)$ that lies on the switching line $x + by = 0$ (but outside the sliding motion region), let $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ denote the states of the system at successive times during which the switching line is crossed, where (x_n, y_n) is the first crossing to fall in the sliding motion region (see Figure 1(c)). Derive a recurrence formula of the form

$$y_{k+1} = T(y_k)$$

showing how each new switching point is obtained from the previous one.

Solution. The points (x_k, y_k) and (x_{k+1}, y_{k+1}) both lie at the intersection of the line $x + by = 0$ and the parabola $x = a + \frac{1}{2}y^2$ (if $x_k > 0$) or the parabola $x = a - \frac{1}{2}y^2$ (if $x_k < 0$) for the appropriate value of a. In the case $x_k > 0$ (therefore $y_k < 0$, from the computation in the solution to part (a) above this shows that

$$y_k = -b - \sqrt{b^2 - 2a},$$
$$y_{k+1} = -b + \sqrt{b^2 - 2a},$$

and in particular $y_k + y_{k+1} = -2b$, or in other words

$$y_{k+1} = -2b - y_k.$$

In the other case $y_k > 0$ one can check analogously that $y_{k+1} = 2b - y_k$. To summarize, this shows that in general we have the recurrence relation

$$y_{k+1} = T(y_k)$$

where

$$T(y) = \begin{cases} -y_k + 2b & \text{if } y_k > 0, \\ -y_k - 2b & \text{if } y_k < 0. \end{cases}$$

(c) Use the answer to part (b) above to find a formula for the number n of times the system undergoes switching (i.e., the number of times the switching line is crossed) before it enters the sliding motion phase, as a function of the initial point (x_0, y_0). Illustrate this formula by applying it in the specific case $b = 0.5, (x_0, y_0) = (-2.2, 4.4)$.

Solution. Let's try the example first. Applying the map T starting from $y_0 = 4.4$ gives a sequence of numbers

$$4.4 \mapsto -3.4 \mapsto 2.4 \mapsto -1.4 \mapsto 0.4$$

The last value 0.4 falls in the sliding motion region $|y| \le b$ so the iteration stops there. In this case the number of switchings is $n = 4$. It is easy to see how to generalize this. In the general case, the y_k's alternate in sign, but if we look only at their absolute values $z_k = |y_k|$ then they satisfy the much simpler recurrence

$$z_{k+1} = z_k - 2b,$$

and the number n is the number of iterations required to bring z_0 to a number in the range $[-b, b]$. This translates to the condition

$$-b < z_0 - n \cdot 2b < b,$$

which can also be written as

$$n \cdot 2b < z_0 + b < (n+1) \cdot 2b.$$

As a consequence, we can express n in terms of z_0 with a formula involving the floor function:

$$n = \left\lfloor \frac{z_0 + b}{2b} \right\rfloor.$$

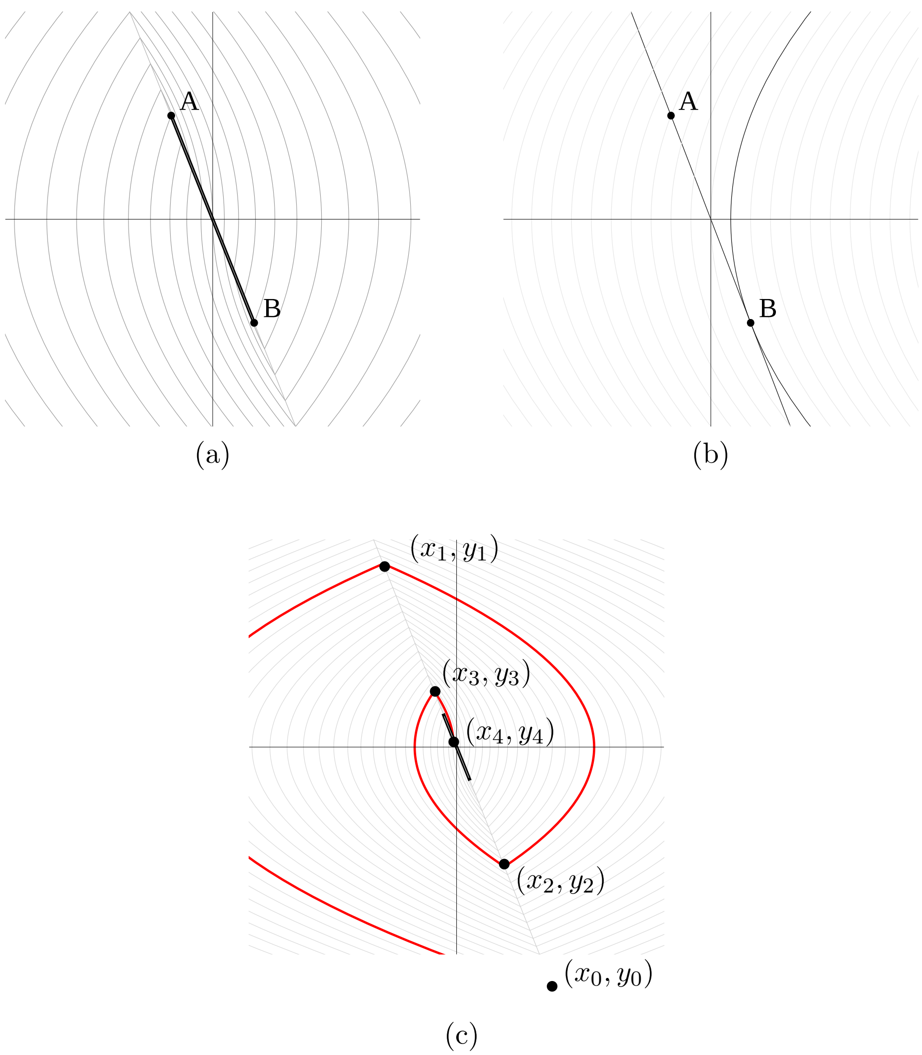


Figure 1: (a) The phase portrait of the switching control and the sliding motion region. (b) At the point B, the curve $x = a + \frac{1}{2}y^2$ becomes tangent to the switching line $x + by = 0$.

3. The optimal switching control rule in the electromagnetic levitation problem leads to the system

$$\dot{x} = y,$$
$$\dot{y} = -\operatorname{sgn}\left(x + \tfrac{1}{2}y|y|\right).$$

Find a formula for the time $\tau(x_0, y_0)$ it takes the system to get to the rest point at the origin from an arbitrary initial state (x_0, y_0) (see the figure below).

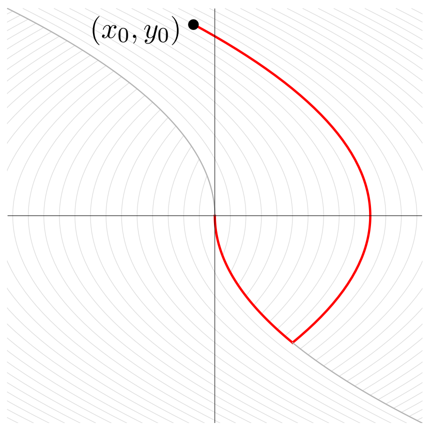


Figure 2: Phase portrait for the optimal switching control.

Solution. We need to divide into two cases according to which side of the curve $x = -\frac{1}{2}y|y|$ the initial point (x_0, y_0) lies on. First, assume that $x_0 > -\frac{1}{2}y_0|y_0|$ (as in the example in the figure above). In this case, the point first flows along the parabola $x = a - \frac{1}{2}y^2$ (where a is determined by the condition $x_0 = a - \frac{1}{2}y_0^2$, giving $a = x_0 + \frac{1}{2}y_0^2$), until it meets the point (x_1, y_1) at the intersection of this parabola with the second parabola $x = \frac{1}{2}y^2$ (more precisely, the half-parabola where $y < 0$—see the figure). So, (x_1, y_1) satisfies the simultaneous equations

$$x_1 = a - \tfrac{1}{2}y_1^2 = \tfrac{1}{2}y_1^2,$$

from which it is easy to find that

$$(x_1, y_1) = \left(\tfrac{1}{2}(x_0 + \tfrac{1}{2}y_0^2), -\sqrt{x_0 + \tfrac{1}{2}y_0^2}\right).$$

From there, the point flows directly to $(x_2, y_2) = (0, 0)$. The total time to get to $(0, 0)$ is given by the sum of the absolute values of the differences of the y-coordinates (since, from the equations of motion, $|\dot{y}| = 1$ always):

$$\tau(x_0, y_0) = |y_0 - y_1| + |y_1 - y_2| = y_0 - 2y_1 = y_0 + 2\sqrt{x_0 + \tfrac{1}{2}y_0^2}.$$

A similar computation for the second case in which $x_0 \le -\frac{1}{2}y_0|y_0|^2$ gives

$$\tau(x_0, y_0) = -y_0 + 2\sqrt{-x_0 + \tfrac{1}{2}y_0^2}.$$

To summarize, the final formula for $\tau(x, y)$ is:

$$\tau(x,y) = \begin{cases} y + 2\sqrt{x + \frac{1}{2}y^2} & \text{if } x > -\frac{1}{2}y|y|, \\ -y + 2\sqrt{-x + \frac{1}{2}y^2} & \text{if } x \le -\frac{1}{2}y|y|. \end{cases}$$