Homework due: Friday 6/8 at 12 pm (in my office, MSB 2218) or Wednesday 6/6 in class

Problems

1. A child standing on a swing bends her knees up and down in a periodic motion. This causes a slight change in the resonant frequency of the swing. In the approximation of small amplitude oscillations, the equation of motion for this system is

$$\ddot{x} = -(\omega^2 \pm \epsilon^2)x = -\omega_\pm^2 x$$

where ϵ is a small number and we use the notation

$$\pm = \begin{cases} +1 & \text{if } \sin(\omega t) > 0, \\ -1 & \text{if } \sin(\omega t) < 0, \end{cases}$$

$$\omega_+ = \sqrt{\omega^2 + \epsilon^2},$$
$$\omega_- = \sqrt{\omega^2 - \epsilon^2}.$$

Note that the frequency of the knee-bending is chosen to coincide with the resonant frequency of the pendulum. The goal of this problem is to show that this causes the rest point at $x = \dot{x} = 0$ to become unstable—a phenomenon known as **parametric resonance** (that children everywhere are grateful for, since it enables them to swing on a swing without the assistance of a parent)[1].

(a) Write an equivalent form of the system as a planar first-order system.

(b) Use reasoning similar to our analysis of the inverted pendulum with an oscillating base to find 2×2 matrices S_+, S_- such that the criterion for stability of the system at the rest point $x = \dot{x} = 0$ can be written as $|\operatorname{tr}(P)| < 2$, where $P = S_- S_+$.

(c) Deduce from part (b) that the condition for stability is

$$\left| 2\cos\left(\frac{\pi\omega_+}{\omega}\right)\cos\left(\frac{\pi\omega_-}{\omega}\right) - \left(\frac{\omega_+}{\omega_-} + \frac{\omega_-}{\omega_+}\right)\sin\left(\frac{\pi\omega_+}{\omega}\right)\sin\left(\frac{\pi\omega_-}{\omega}\right) \right| < 2$$

(d) Define a new variable $s = \epsilon^2/\omega^2$, and show that the condition above translates to checking that $|F(s)| < 2$, where

$$\begin{aligned} F(s) = {} & 2\cos\left(\pi\sqrt{1+s}\right)\cos\left(\pi\sqrt{1-s}\right) \\ & - \left(\frac{\sqrt{1+s}}{\sqrt{1-s}} + \frac{\sqrt{1-s}}{\sqrt{1+s}}\right)\sin\left(\pi\sqrt{1+s}\right)\sin\left(\pi\sqrt{1-s}\right). \end{aligned}$$

(e) Use a computer or graphing calculator to plot the graph of $F(s)$, and verify that the inequality $|F(s)| > 2$ holds for all $0 < s < 1$. Conclude that "parametrically resonated swings" are unstable (and therefore fun!).

[1] See the Wikipedia article `http://en.wikipedia.org/wiki/Parametric_oscillator`.

2. In the two-dimensional phase portrait of the switching control scheme

$$\dot{x} = y,$$
$$\dot{y} = -\operatorname{sgn}(x + by)$$

that arises in connection with the electromagnetic levitation problem (where b is a positive numerical parameter), the system will go into a **sliding motion** (a.k.a. **chattering**) phase after reaching the line segment AB of the switching line $x+by = 0$ shown in Figure 1(a) below. The sliding motion is characterized by the property that the vector field of the equation on both sides of the switching line pushes the particle back towards the line.

(a) Find the coordinates of the endpoints A and B of the sliding motion segment.

Hint. See Figure 1(b) and its caption below.

(b) If we start the system at a point $(x_0, y_0) = (-by_0, y_0)$ that lies on the switching line $x + by = 0$ (but outside the sliding motion region), let $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ denote the states of the system at successive times during which the switching line is crossed, where (x_n, y_n) is the first crossing to fall in the sliding motion region (see Figure 1(c)). Derive a recurrence formula of the form

$$y_{k+1} = T(y_k)$$

showing how each new switching point is obtained from the previous one.

(c) Use the answer to part (b) above to find a formula for the number n of times the system undergoes switching (i.e., the number of times the switching line is crossed) before it enters the sliding motion phase, as a function of the initial point (x_0, y_0). Illustrate this formula by applying it in the specific case $b = 0.5, (x_0, y_0) = (-2.2, 4.4)$.

Hint. Try the numerical example first, then generalize the idea. Before writing a formula think of a verbal description of how the number n is computed from z_0, then try translating it into a formula. The formula may involve the floor function $\lfloor x \rfloor$ (which returns the largest integer that is $\leq x$).

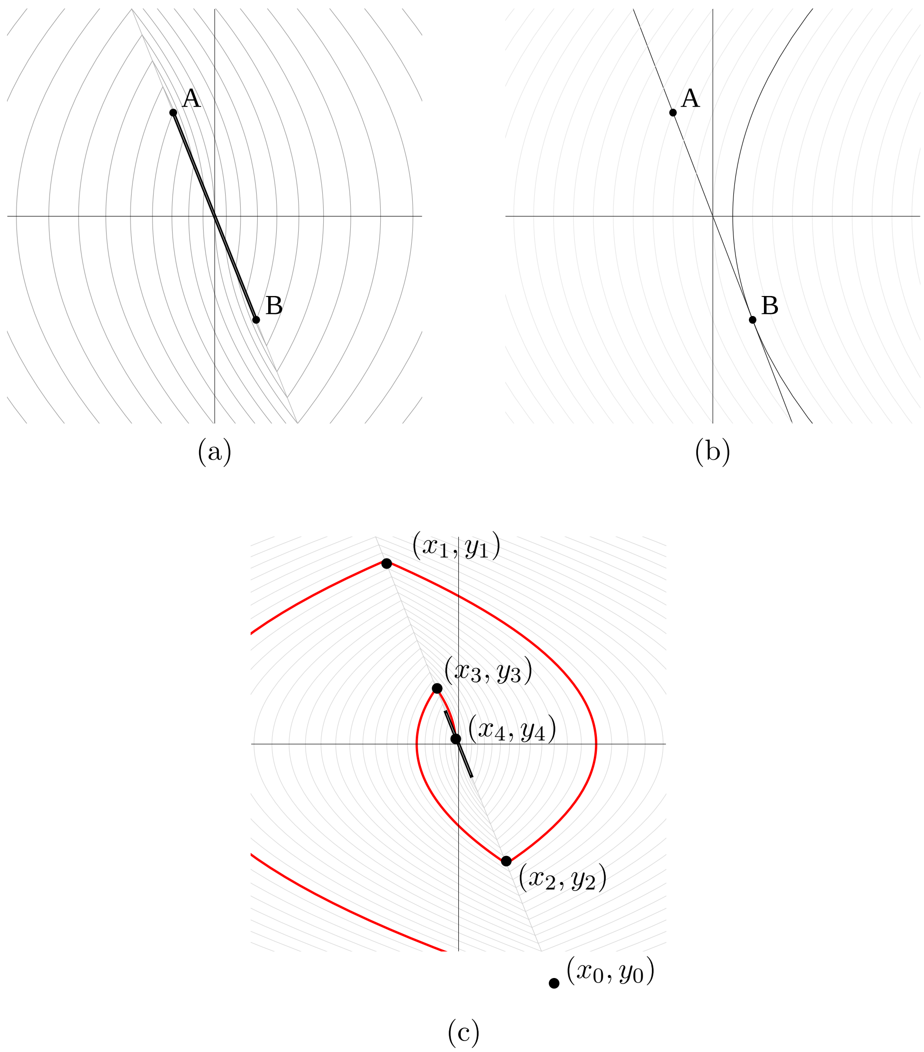

Figure 1: (a) The phase portrait of the switching control and the sliding motion region. (b) At the point B, the curve $x = a + \frac{1}{2}y^2$ becomes tangent to the switching line $x + by = 0$.

3. The optimal switching control rule in the electromagnetic levitation problem leads to the system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\operatorname{sgn}\left(x + \tfrac{1}{2}y|y|\right). \end{aligned}$$

Find a formula for the time $\tau(x_0, y_0)$ it takes the system to get to the rest point at the origin from an arbitrary initial state (x_0, y_0) (see the figure below).

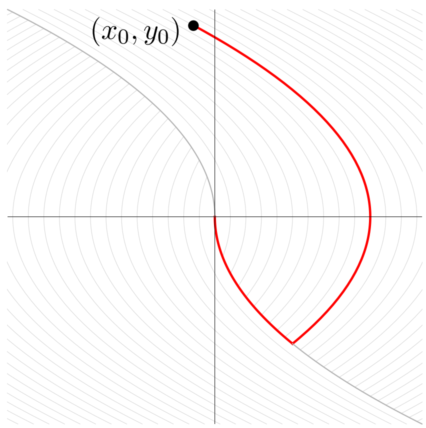

Figure 2: Phase portrait for the optimal switching control.