Question 1

Define the functional

$$\Phi(q) = \int_0^{\pi/2} (\dot{q}(t)^2 - q(t)^2)\, dt$$

Find the function q where $\Phi(q)$ is minimal, subject to the boundary conditions $q(0) = 0, q(\pi/2) = 3$.

Solution. The functional is defined in terms of the Lagrangian $L(\dot{q}, q) = \dot{q}^2 - q^2$. The Euler-Lagrange equation for this Lagrangian is

$$2\ddot{q} = \frac{d}{dt}(2\dot{q}) = -2q,$$

or $\ddot{q} = -q$. The general solution has the form

$$q(t) = A\cos(t) + B\sin(t).$$

The boundary conditions give $q(0) = A = 0$, $q(\pi/2) = B\sin(\pi/2) = 3$, i.e., $B = 3$, so the solution to the minimization problem is

$$q(t) = 3\sin(t).$$

Question 2

Find a necessary and suffficient condition for the linear planar system

$$\dot{p} = ap + bq$$
$$\dot{q} = cp + dq$$

to be Hamiltonian, in terms of the coefficients a, b, c, d. In the case when it is Hamiltonian, find a formula for the Hamiltonian $H(p, q)$.

Solution. The necessary and sufficient condition is that the divergence of the vector field $(F, G) = (ap + bq, cp + dq)$ should be zero. This translates to the condition

$$\frac{\partial F}{\partial p} + \frac{\partial G}{\partial q} = a + d = 0,$$

i.e., the system is Hamiltonian if and only if $d = -a$.

In the case when $d = -a$, the Hamiltonian $H(p, q)$ needs to satisfy

$$-\frac{\partial H}{\partial q} = ap + bq,$$
$$\frac{\partial H}{\partial p} = cp - aq.$$

Integrating the first equation with respect to q gives

$$H(p, q) = -apq - \tfrac{1}{2}bq^2 + g(p)$$

where $g(p)$ is an arbitrary function. Plugging this expression into the second equation gives

$$cp - aq = \frac{\partial H}{\partial p} = -aq + g'(p),$$

which gives $g'(p) = cp$, or $g(p) = \frac{1}{2}cp^2$ (plus an arbitrary integration constant, which we choose to be 0). Thus, we have found the candidate Hamiltonian

$$H(p, q) = \tfrac{1}{2}cp^2 - \tfrac{1}{2}bq^2 - apq.$$

Although we know this procedure must yield the correct answer, it is good practice (and in this case very easy) to verify that $H(p, q)$ satisfies the desired equations $-\frac{\partial H}{\partial q} = ap + bq$, $\frac{\partial H}{\partial p} = cp - aq$.

Question 3

You are given the second-order equation $\ddot{x} = -x^3$. Find a conserved quantity $E(\dot{x}, x)$.

Solution. In a general conservative system $\ddot{x} = -U'(x)$, the total energy (which becomes the Hamiltonian when expressed in the p and q coordinates) is conserved. It is the sum of the kinetic and potential energies:

$$E = \tfrac{1}{2}\dot{x}^2 + U(x).$$

In this case $U(x) = \frac{1}{4}x^4$, so the conserved quantity is

$$E = \tfrac{1}{2}\dot{x}^2 + \tfrac{1}{4}x^4.$$

This can be easily verified by differentiation:

$$\frac{dE}{dt} = \dot{x}\ddot{x} + x^3\dot{x} = \dot{x}(\ddot{x} + x^3) = \dot{x} \cdot 0 = 0.$$